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# Stochastic Geometry, Spatial Statistics and Random Fields

## Models and Algorithms

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# Preface

This volume is an attempt to provide a graduate level introduction to various aspects of stochastic geometry, spatial statistics and random fields, with special emphasis placed on fundamental classes of models and algorithms as well as on their applications. This book has a strong focus on simulations and includes extensive code in *Matlab* and *R*, which are widely used in the mathematical community. It can be seen as a continuation of the recent volume 2068 of Lecture Notes in Mathematics, where other issues of stochastic geometry, spatial statistics and random fields were considered with a focus on asymptotic methods.

The present volume comprises selected contributions to the Summer Academy on Stochastic Analysis, Modelling and Simulation of Complex Structures (cf. http://www.uni-ulm.de/mawi/summer-academy-2011.html) which took place during September 11-24, 2011 at the Söllerhaus, an Alpine conference center of the University of Stuttgart and RWTH Aachen, in the village of Hirschegg (Kleinwalsertal). It was organized by the Institute of Stochastics of Ulm University. In contrast with previous summer schools on this subject (Sandbjerg 2000, Martina Franca 2004, Sandbjerg 2007, Hirschegg 2009), the focus of this summer school was on models and algorithms from stochastic geometry, spatial statistics and random fields which can be used for the analysis, modelling and simulation of geometrically complex microstructures. Examples of such microstructures are complex point patterns and networks which appear in advanced functional materials and whose geometrical structures are closely related to the (macroscopic) physical properties of the underlying technical and biological materials.

This summer school hosted 43 young participants from 8 countries (Australia, Czech Republic, Denmark, France, Germany, Russia, Switzerland, UK). Fourteen international experts gave lectures on various aspects of stochastic geometry, spatial statistics and random fields. In addition, students and young researchers were able to give their own short talks.

As stated above, this volume is focused on fundamental classes of models and algorithms from stochastic geometry, spatial statistics and random fields as well as on their applications to the analysis, modelling and simulation of geometrically

complex objects like points patterns, tessellations, graphs, and trees. It reflects recent progress in these areas, with respect to both theory and applications.

Point processes play an especially important role in many parts of the book. Poisson processes, the most fundamental class of point processes, are considered in Chapter 1 as approximations of other (not necessarily Poisson) point processes. In Chapter 2, they are used as a basic reference model when comparing the clustering properties of various point processes and, in Chapters 3 and 5, they are used in order to construct some basic classes of random tessellations and Boolean random functions, respectively. Several further classes of point processes considered in this book can be seen as generalizations of Poisson processes and are derived in various ways from Poisson processes. For example, Gibbs processes (Chapters 1 and 2), Poisson cluster processes (Chapters 2, 4 and 13), Cox processes (Chapters 2, 4, 7 and 13), and hard-core processes (Chapters 2 and 4).

From another perspective, point processes can be seen as a special class of random closed sets, which are also fundamental objects in stochastic geometry. In addition to point processes, a number of other classes of random closed sets are considered in the book; in particular, germ-grain models (Chapters 2, 4 and 5), random fiber processes (Chapters 4 and 6), random surface processes (Chapter 5), random tessellations (Chapters 3 and 6), random spatial networks (Chapters 2 and 4) and isotropic convex bodies (Chapter 8). Random marked closed sets (Chapter 6) – in particular, random marked point processes (Chapter 13) – also play an important role in the book.

Another focus of the book is on various aspects of random fields. In Chapter 5, a class of random fields is considered which can be seen as a generalization of random closed sets and, in particular, of germ-grain models. In Chapter 6, random fields are used in order to construct random marked sets. Some basic ideas of principal component analysis for random fields are discussed in Chapter 9. Genetic models involving random fields are considered in Chapter 10. Chapter 11 deals with extrapolation techniques for two large classes of random fields: square-integrable stationary random fields and stable random fields. In addition, various simulation algorithms for random fields are discussed in Chapters 12 and 13, in particular for Gaussian Markov random fields, fractional Gaussian fields, spatial Lévy processes and random walks.

The book is organized as follows. The first four chapters deal with point processes, random tessellations and random spatial networks, with a number of different examples of their applications. Chapter 1 gives an introduction to Stein's method, a powerful technique for computing explicit error bounds for distributional approximation. The classical case of normal approximation is used as an initial motivation. Then, the main part of the chapter is devoted to presenting the key concepts of Stein's method in a much more general framework, where the approximating distribution and the space it lives on can be almost arbitrary. This is particularly appealing for distributional approximation of various point-process models considered in stochastic geometry and spatial statistics. Chapter 2 reviews examples, methods, and recent results concerning the comparison of clustering properties of point processes. The approach is founded on the observation that void probabilities and

moment measures can be used as two complementary tools for capturing clustering phenomena in point processes. Various global and local functionals of random geometric models driven by point processes are considered which admit more or less explicit bounds involving void probabilities and moment measures. Directional convex ordering of point processes is also discussed. Such an ordering turns out to be an appropriate choice, combined with the notion of (positive or negative) association, when comparison to the Poisson point process is considered. Chapter 3 introduces various tessellation models and discusses their application as models for cellular materials. First, the notion of a random tessellation is introduced, along with the most well-known model types (Voronoi and Laguerre tessellations, hyperplane tessellations, STIT tessellations), and their basic geometric characteristics. Assuming that a cellular material is a realization of a suitable random tessellation model, characteristics of these models can be estimated from 3D images of the material. An explanation is given of how such estimates are obtained and how they can be used to fit tessellation models to the observed microstructure. In Chapter 4, three classes of stochastic morphology models are presented. These describe different microstructures of functional materials by means of methods from stochastic geometry, graph theory and time series analysis. The structures of these materials strongly differ from one another. In particular, the following are considered: organic solar cells, which are anisotropic composites of two materials; nonwoven gas-diffusion layers in proton exchange membrane fuel cells, which consist of a system of curved carbon fibers; and, graphite electrodes in Li-ion batteries, which are an isotropic two-phase system (i.e., consisting of a pore and a solid phase). The goal of this chapter is to give an overview of how models from stochastic geometry, graph theory and time series analysis can be applied to the stochastic modeling of functional materials and how these models can be used for material optimization with respect to functionality.

The three following chapters deal with Boolean random functions, random marked sets and space-time models in stochastic geometry. In Chapter 5, the notion of Boolean random functions is considered. These are generalizations of Boolean random closed sets. Their construction is based on the combination of a sequence of primary random functions using the operations of supremum or infimum. Their main properties are given in the case of scalar random functions built on Poisson point processes. Examples of applications to the modeling of rough surfaces are also given. In Chapter 6, random marked closed sets are investigated. Special models with integer Hausdorff dimension are presented based on tessellations and numerical solutions of stochastic differential equations. Statistical analysis is developed which involves the random-field model test and estimation of first and second order characteristics. Real data analyses from neuroscience (track modeling marked by spiking intensity) and materials research (grain microstructure with disorientations of faces) are presented. Dimension reduction of point processes with Gaussian random fields as covariates, a recent development, is generalized in three different ways. Chapter 7 deals with space-time models in stochastic geometry which are used in many applications. Most such models are space-time point processes. Other common models are based on growth models of random sets. This chapter aims to

present more general models, where time is considered to be either discrete or continuous. In the discrete-time case the authors focus on state-space models and the use of Monte Carlo methods for the inference of model parameters. Two applications to real situations are presented: evaluation of a neurophysiological experiment and models of interacting discs. In the continuous-time case, the authors discuss space-time Lévy-driven Cox processes with different second-order structures.

The following four chapters are devoted to different issues of spatial statistics and random fields. Chapter 8 contains an introduction to rotational integral geometry. This is the key tool in local stereological procedures for estimating quantitative properties of spatial structures. In rotational integral geometry, the focus is on integrals of geometric functionals with respect to rotation invariant measures. Rotational integrals of intrinsic volumes are studied. The opposite problem of expressing intrinsic volumes as rotational integrals is also considered. An explanation is given of how intrinsic volumes can be expressed as integrals with respect to geometric functionals defined on lower dimensional linear subspaces. The rotational integral geometry of Minkowski tensors is briefly discussed as well as a principal rotational formula. These tools are then applied in local stereology leading to unbiased stereological estimators of mean intrinsic volumes for isotropic random sets. At the end of the chapter, emphasis is put on how these procedures can be implemented when automatic image analysis is available. Chapter 9 gives an introduction to the methods of functional data analysis. The authors present the basics from principal component analysis for functional data together with the functional analytic background as well as the data analytic counterpart. As prerequisites, they give an introduction into presentation techniques for functional data and some smoothing techniques. In Chapter 10, a challenging statistical problem in modern genetics is considered: how to identify the collection of factors responsible for increasing the risk of specified complex diseases. Enormous progress in the field of genetics has made possible the collection of very large genetic datasets for analysis by means of various complementary statistical tools. Thus, one has to operate with data of huge dimensions and this is one of the main difficulties in detection of genetic susceptibility to common diseases such as hypertension, myocardial infarction and others. In this chapter, the author concentrates on the multifactor dimensionality reduction method. Modifications and extensions are also discussed. Recent results on the central limit theorem related to this method are provided as well. In addition, the main features of logistic regression are discussed and simulated annealing for stochastic minimization of functions defined on a graph with forests as vertices is tackled. Chapter 11 introduces basic statistical methods for the extrapolation of stationary random fields. The problem of extrapolation (prediction) of random fields arises in geosciences, mining, oil exploration, hydrosciences, insurance, and many other fields. The techniques to solve this problem are fundamental tools in geostatistics that provide statistical inference for spatially referenced variables of interest. Examples of such quantities are the amount of rainfall, concentration of minerals and vegetation, soil texture, population density, economic wealth, storm insurance claim amounts, etc. For square integrable fields, kriging extrapolation techniques are considered. For (non–Gaussian) stable fields, which are known to be heavy-tailed, further extrapo-

lation methods are described and their properties are discussed. Two of them can be seen as direct generalizations of kriging.

The book concludes with two chapters which deal with algorithms for Monte Carlo simulation of random fields. The generation of random spatial data on a computer is an important tool for understanding the behavior of spatial processes. Chapter 12 describes how to generate realizations of the main types of spatial processes, including Gaussian and Markov random fields, point processes (including the Poisson, compound Poisson, cluster, and Cox processes), spatial Wiener processes, and Lévy fields. Concrete *Matlab* code is also provided. The purpose of Chapter 13 is to exemplify construction of selected coupling-from-the-past algorithms, using simple examples and discussing code which can be run in the statistical scripting language *R*. The simple examples are the symmetric random walk with two reflecting boundaries; a very basic continuous state-space Markov chain; the Ising model with external field; and, a random walk with negative drift and a reflecting boundary at the origin. In parallel with this, a discussion is given of the relationship between coupling-from-the-past algorithms on the one hand, and uniform and geometric ergodicity on the other.

The authors of this book have tried to present many different methods developed in various fields of stochastic geometry, spatial statistics and random fields that merit communication to a broader audience. All chapters contain introductory sections which are easily accessible for non-specialists who want to become acquainted with modern techniques of stochastic geometry, spatial statistics and random fields. New results, which have been obtained only recently, are also presented. Each chapter provides a number of exercises which will help the reader to use the stochastic models and algorithms considered in this book autonomously.

Ulm *Volker Schmidt*
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# Acronyms

## Sets

| | |
|---|---|
| $\mathbb{C}$ | complex numbers |
| $\mathbb{N}$ | positive integer numbers |
| $\mathbb{Q}$ | rational numbers |
| $\mathbb{R}$ | real numbers |
| $\mathbb{Z}$ | all integer numbers |
| $\mathsf{G}$ | class of all closed sets |
| $\mathcal{C}$ | class of all compact sets |
| $\mathcal{K}^d_{conv}$ | class of all convex bodies in $\mathbb{R}^d$ |
| $\mathcal{E}^d_k$ | family of affine $k$-planes in $\mathbb{R}^d$ |
| $\mathcal{B}_0(\mathbb{R}^d)$ | family of bounded Borel sets in $\mathbb{R}^d$ |

## Probability theory

| | |
|---|---|
| $\xi, \eta$ | random variables |
| $X, Y$ | random sets |
| $\mathbf{P}_\xi$ | probability measure of $\xi$ |
| $\mathbf{E}\,\xi$ | expectation of $\xi$ |
| $\mathbf{corr}(\xi, \eta)$ | correlation of $\xi$ and $\eta$ |
| $\mathbf{cov}(\xi, \eta)$ | covariance of $\xi$ and $\eta$ |
| $\mathbf{var}\,\xi$ | variance of $\xi$ |
| $\mathsf{N}$ | set of all locally finite simple point patterns |
| $\mathbf{1}$ | indicator function |
| $\mathrm{Pois}(\lambda)$ | Poisson distribution with parameter $\lambda$ |
| $\mathrm{Exp}(\lambda)$ | Exponential distribution with parameter $\lambda$ |
| $\mathrm{Ber}(p)$ | Bernoulli distribution with parameter $p$ |
| $\mathrm{Binom}(n, p)$ | Binomial distribution with parameters $n$ and $p$ |

**Other notations**

| | |
|---|---|
| $\mathsf{E}$ | energy |
| $\mathtt{T}$ | tessellation |
| diag | diagonal of a matrix |
| dist | distance function |
| conv | convex hull |
| supp | support |
| card | cardinality of a set |
| diam | diameter |
| $\mathsf{SO}$ | rotation group |
| $\Im z$ | imaginary part of a complex number |
| $\Re z$ | real part of a complex number |
| sgn | signum function |
| Lip | Lipschitz operator |
| span | linear hull |
| $\nu_d$ | Lebesgue measure |
| $V_0,\dots,V_d$ | intrinsic volumes |
| $\chi$ | Euler characteristic |
| $K^r$ | $r$-neighbourhood of $K$ |
| $\mathrm{Int}\, K$ | interior of $K$ |
| $K\|L$ | orthogonal projection of $K$ onto $L$ |
| $B_r(o)$ | ball of radius $r$ centred in the origin |
| $S\alpha S$ | symmetric $\alpha$-stable |