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Symmetric Spaces and the Kashiwara-Vergne Method

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Pour Anne-Marie

Preface

Let G/K denote a Riemannian symmetric space of the noncompact type.

> The action of the G-invariant differential operators on G/K on the radial functions on G/K is isomorphic with the action of certain differential operators with constant coefficients. The isomorphism in question is used in Harish-Chandra's work on the Fourier analysis on G and is related to the Radon transform on G/K.
> For the case when G is complex a more direct isomorphism of this type is given, again as a consequence of results of Harish-Chandra.

This is a quote from Sigurður Helgason, *Fundamental solutions of invariant differential operators on symmetric spaces*, Am. J. Math. 86, p. 566 (1964), one of the first mathematical papers I have studied. As a consequence he showed that every (non-zero) G-invariant differential operator D on G/K has a fundamental solution, whence the solvability of the differential equation $Du = f$. Those results were fascinating to me, they still are and they fostered my interest in invariant differential operators as well as Radon transforms. Yet I was dreaming simpler proofs could be given, without relying on Harish-Chandra's deep study of semisimple Lie groups...

In the beginning of 1977 Michel Duflo [18] gave an analytic interpretation of the celebrated *Duflo isomorphism* he had exhibited 6 years before by means of algebraic constructions in an enveloping algebra. As a consequence he proved that every (non-zero) bi-invariant differential operator on a Lie group has a local fundamental solution and is locally solvable. His work used delicate analysis on the group.

Then, in the fall of 1977, came a preprint by Masaki Kashiwara and Michèle Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions* [30], showing that similar (and even stronger) results could be obtained—for solvable Lie groups at least—by means of "elementary", but very clever, formal computations with the exponential mapping and the Campbell-Hausdorff formula only.

After explaining this method for Lie groups (with updates from the 2008–2009 papers by Anton Alekseev and Charles Torossian) I will give here a detailed account of its extension to general symmetric spaces. The present text is an attempt at a self-contained monograph on the Kashiwara-Vergne approach to invariant analysis. This rewritten and updated version of my former papers [43–47] includes several

unpublished results and a few open questions. It is also a greatly expanded version of a talk given at the 2007 Reykjavik conference in honor of Helgason's eightieth birthday.

Acknowledgements

Sigurður Helgason's influence on my own work goes far beyond the above quote of course. His clearly written books and articles have been a constant reference and source of inspiration to me for many years and, above all, his friendly advice on many occasions has been extremely helpful. Thank you so much Sigurður.

I am indebted to Mogens Flensted-Jensen for several stimulating discussions, particularly about the rank one case.

I am very thankful to Anton Alekseev, Charles Torossian and Michèle Vergne for enlightening explanations of their work.

Nice, France François Rouvière

Contents

Notation

1 General Notation

The sign := indicates a definition.

$\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of natural integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the ring of all integers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

$\operatorname{sh} x = (e^x - e^{-x})/2$, $\operatorname{ch} x = (e^x + e^{-x})/2$, $\operatorname{th} x = \operatorname{sh} x / \operatorname{ch} x$ and $\coth x = \operatorname{ch} x / \operatorname{sh} x$ are the classical hyperbolic functions.

In Chaps. 1 and 4 we shall use three commuting involutions acting on a function f of two variables in a vector space

$$f^{\vee}(X, Y) := f(-X, -Y)\,,\ \overline{f}(X, Y) := f(Y, X)\,,\ \tilde{f}(X, Y) := f(-Y, -X).$$

Dots are used with several meanings, which should be clear from their context: if a group G acts on a set, the action of $g \in G$ transforms a point x of the set into $g \cdot x$.

$\langle ., . \rangle$ are duality brackets between a vector space and its dual, or sometimes denote a scalar product. ^{t} means transpose.

$\operatorname{tr}_V u$ denotes the trace of an endomorphism u of a finite dimensional vector space V and $\det_V u$ its determinant. The subscripts V may be dropped when no confusion arises.

Smooth means C^∞ and supp denotes the support.

$\mathcal{D}(M)$ denotes the space of **test functions** on a manifold M, i.e. smooth complex-valued compactly supported functions on M, equipped with the Schwartz topology (see e.g. [28, p. 239]). Its dual $\mathcal{D}'(M)$ is the space of distributions on M.

If f maps a manifold into another, $D_x f$ is the tangent map (differential) of f at x. In case of several variables x, y etc., $\partial_x f$ is the partial differential with respect to x.

Let us recall the classical multi-index notation when taking coordinates $(x_1, \ldots, x_n)$ in a vector space: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \ , \ \alpha! = \alpha_1! \cdots \alpha_n!$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ , \ \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \ , \ \partial_i = \partial/\partial x_i \, .$$

2 Lie Groups and Lie Algebras

The identity element of a Lie group G is denoted by e (not to be confused, of course, with a notation such as e^X or $e(X, Y)$).

L_g, resp. R_g, is the left, resp. right, translation in G, that is $L_g x := gx$ and $R_g x := xg$ for $g, x \in G$.

For X, Y in the Lie algebra $\mathfrak{g}$ of G, with bracket $[.,.]$, we write $\operatorname{ad} X(Y) := [X, Y]$. The notation $x := \operatorname{ad} X$, $y := \operatorname{ad} Y$ will be frequently used in Chaps. 1 and 4; for example, $xy^2xY = [X, [Y, [Y, [X, Y]]]]$, etc.

If $\mathfrak{a}, \mathfrak{b}$ are vector subspaces of $\mathfrak{g}$, $[\mathfrak{a}, \mathfrak{b}]$ is the space of all finite sums $\sum_i [A_i, B_i]$ with $A_i \in \mathfrak{a}$, $B_i \in \mathfrak{b}$.

The adjoint representation Ad of G on $\mathfrak{g}$ will most often be denoted by a dot: $g \cdot X := \operatorname{Ad} g(X)$ for $g \in G$, $X \in \mathfrak{g}$.

The exponential mapping $\exp : \mathfrak{g} \to G$ will most often be written as $X \mapsto \exp X = e^X$; thus $e^{g \cdot X} = g e^X g^{-1}$ and $\operatorname{Ad} e^X = e^{\operatorname{ad} X}$.

The Campbell-Hausdorff formula expands $V(X, Y) := \log\left(e^X e^Y\right)$ as a series of brackets of X and Y.

The differential at $X \in \mathfrak{g}$ of the exponential mapping and its Jacobian are

$$D_X \exp = D_e L_{e^X} \circ \frac{1 - e^{-x}}{x} \ , \ j(X) = \det{}_{\mathfrak{g}} \frac{1 - e^{-x}}{x} \tag{1}$$

with $x = \operatorname{ad} X$.

Caution: this notation j is only used in Chap. 1. In Chaps. 3 and 4, j has a more general meaning (Definition 3.2).

3 Symmetric Lie Algebras

A **symmetric Lie algebra** is a couple $(\mathfrak{g}, \sigma)$ where $\mathfrak{g}$ is a (finite dimensional real) Lie algebra and σ is an involutive automorphism of $\mathfrak{g}$. If $\mathfrak{h}$, resp. $\mathfrak{s}$, denotes the $+1$, resp. -1, eigenspace of σ in $\mathfrak{g}$, we have the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ with $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s}$, $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h}$ by the rule of signs.

The subspace $\mathfrak{h}_* := [\mathfrak{s}, \mathfrak{s}]$ is an ideal of $\mathfrak{h}$.

The subspace $\mathfrak{s}$ is not in general a Lie algebra, but inherits from $\mathfrak{g}$ a structure of **Lie triple system**: denoting by $L(X,Y)$ the map $Z \mapsto [X,Y,Z] := [[X,Y],Z]$ from $\mathfrak{s}$ into itself, we have

(i) $L(X,X) = 0$
(ii) $[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0$
(iii) $L(X,Y)$ is a derivation of the trilinear product $[.,.,.]$, that is

$$L(X,Y)[U,V,W] = [L(X,Y)U,V,W] + [U,L(X,Y)V,W] + [U,V,L(X,Y)W]$$

for all $X,Y,Z,U,V,W \in \mathfrak{s}$ (see [37, p. 78]).

4 Symmetric Spaces

Throughout the text G denotes a connected real Lie group and H a closed subgroup of G. Let $S = G/H$ be the homogeneous space $S = G/H$ of left cosets gH, with $g \in G$. It will be convenient to assume S **is simply connected**; this implies H **is connected**.

Let $\tilde{G}$ be the universal covering of G with canonical projection $p : \tilde{G} \to G$ and $\tilde{H} = p^{-1}(H)$. From the simple connectedness of G/H it follows that $\tilde{H}$ is connected and $\tilde{G}/\tilde{H} = G/H$. Thus, whenever necessary, the group G may be assumed to be simply connected too. These topological properties will only be useful to give a precise definition of the domains we are working on.

The natural action of G on S is denoted by $\tau(g)(g'H) = g \cdot g'H := gg'H$ for $g, g' \in G$, and $o = eH$ is taken as the origin of S.

A homogeneous space $S = G/H$ is a **symmetric space** if G is equipped with an involutive automorphism σ and H lies between the fixed point subgroup of σ in G and its identity component (if connected, H therefore equals this component). We still denote by σ the corresponding automorphism of the Lie algebra $\mathfrak{g}$ of G, whence a symmetric Lie algebra $(\mathfrak{g}, \sigma)$. In the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{s}$ identifies with the tangent space to S at o. We denote by H_* the connected Lie subgroup of H with Lie algebra $\mathfrak{h}_* = [\mathfrak{s}, \mathfrak{s}]$.

The exponential mapping given by the canonical connection of the symmetric space is $\operatorname{Exp} X = e^X H$, $X \in \mathfrak{s}$, where $X \mapsto e^X$ is the exponential mapping of the group G. For $h \in H$ and $X \in \mathfrak{s}$ we have $h \cdot \operatorname{Exp} X = \operatorname{Exp}(h \cdot X)$.

For X, Y in a neighborhood of the origin in $\mathfrak{s}$, the element $Z(X,Y)$ of $\mathfrak{s}$ defined by $\operatorname{Exp} Z(X,Y) = \exp X \cdot \operatorname{Exp} Y$ is the symmetric space analog of the Campbell-Hausdorff element $V(X,Y)$ of Lie groups.

The differential at $X \in \mathfrak{s}$ of the exponential mapping and its Jacobian are, with $x = \operatorname{ad} X$,

$$D_X \operatorname{Exp} = D_o \tau(e^X) \circ \frac{\operatorname{sh} x}{x}\,,\ J(X) = \det{}_{\mathfrak{s}} \frac{\operatorname{sh} x}{x}. \tag{2}$$

The symmetric space S admits a G-invariant measure dx if and only if the Lebesgue measure dX of $\mathfrak{s}$ is H-invariant, that is $|\det_{\mathfrak{s}} \operatorname{Ad} h| = 1$ for all $h \in H$. No absolute value is needed if H is connected. The measures can then be normalized so that

$$\int_S f(x)dx = \int_{\mathfrak{s}} f(\operatorname{Exp} X)J(X)dX$$

if supp f is contained in a suitable neighborhood of the origin.

The superscript H on a space denotes the subspace of H-invariant elements. For example, if U is an H-invariant open subset of S, $\mathcal{D}(U)^H$ is the space of test functions f such that supp $f \subset U$ and $f \circ \tau(h) = f$ for all $h \in H$. Similarly $\mathcal{D}'(U)^H$ is the space of distributions T on U such that $\langle T, f\rangle = \langle T, f \circ \tau(h)\rangle$ for all $f \in \mathcal{D}(U)$, $h \in H$.

$\mathbb{D}(\mathfrak{s}) = S(\mathfrak{s})$ is the algebra of linear differential operators with constant (complex) coefficients on $\mathfrak{s}$, canonically identified with the (complexified) symmetric algebra of $\mathfrak{s}$, and $\mathbb{D}(\mathfrak{s})^H = S(\mathfrak{s})^H$ is the subalgebra of H-invariant operators.

$\mathbb{D}(S)$ denotes the algebra of G-invariant linear differential operators on S.

The specific notation for **line bundles** is explained in Sect. 2.2.

5 Semisimple Notation

In Proposition 3.7, Sects. 3.6, 3.7 and the Appendix, we narrow our investigation to the case of a **Riemannian symmetric space of the noncompact type**. We shall then replace the above general notation by the classical semisimple notation as used in Helgason's books [27–29], to which we refer for details and proofs.

The symmetric space is then $S = G/K$, simply connected, where G is a connected noncompact real semisimple Lie group with finite center and K is a maximal compact subgroup; K is connected. The involutive automorphism is the Cartan involution θ giving the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. For $X, Y \in \mathfrak{g}$ let $\langle X, Y\rangle := -B(X, \theta Y)$, where $B(X, Y) := \operatorname{tr}_{\mathfrak{g}} (\operatorname{ad} X \operatorname{ad} Y)$ is the **Killing form**. The Riemannian structure on S is the G-invariant metric defined by the scalar product $\langle ., .\rangle$ restricted to $\mathfrak{p}$. The exponential mapping Exp is then a global diffeomorphism of $\mathfrak{p}$ onto S.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ its dual space; the dimension of $\mathfrak{a}$ is the **rank** of G/K. A linear form $\alpha \in \mathfrak{a}^*$ is called a **root** of $(\mathfrak{g}, \mathfrak{a})$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, where

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

The dimension $m_\alpha = \dim \mathfrak{g}_\alpha$ is the **multiplicity** of the root α. A point $H \in \mathfrak{a}$ is called **regular** if $\alpha(H) \neq 0$ for all roots α. The set $\mathfrak{a}'$ of regular elements has (finitely many) connected components, the Weyl chambers. Having picked one of them, called the **positive Weyl chamber** and denoted by $\mathfrak{a}^+$ (with closure $\overline{\mathfrak{a}^+}$ in $\mathfrak{g}$),

we say a root is **positive** if it takes positive values on $\mathfrak{a}^+$. Let $\rho := \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha \in \mathfrak{a}^*$ (sum over the set of positive roots α). Let M, resp. M', be the centralizer, resp. normalizer, of $\mathfrak{a}$ in K. They are compact subgroups with the same Lie algebra. The quotient group $W := M'/M$ is a finite group called the **Weyl group**; it acts simply transitively on the set of Weyl chambers.

The choice of $\mathfrak{a}$ gives rise to the **Iwasawa decomposition** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$. The corresponding Lie subgroups of G give the decomposition $G = KAN$, the map $(k, a, n) \mapsto g = kan$ being a diffeomorphism of $K \times A \times N$ onto G.

Introduction

The 1978 Inventiones paper [30] by Kashiwara and Vergne was a breakthrough in the field of analysis on Lie groups. Before it, the proof of several significant results (notably results involving invariant differential operators, their local solvability, Duflo's isomorphism,...) required a detailed knowledge of the structure of the group and its Lie algebra, the use of specific subgroups, etc. Their work opened a new way, only using skillful formal manipulations of Lie brackets, thus providing a direct link to the infinitesimal structure of the group. They carried out their program for solvable Lie groups and conjectured two general identities allowing an extension of the method to arbitrary Lie groups.

Two natural questions (at least) arose from their paper:

- prove the Kashiwara-Vergne conjecture, implying that their method is actually valid for all Lie groups
- investigate possible extensions of the method to symmetric spaces.

A complete solution to the former question kept us waiting for nearly 30 years. The conjecture is now a theorem, proved in full generality by two different methods in [5] resp. [7], by Alekseev and Meinrenken resp. Alekseev and Torossian; see the Notes of Chap. 1 for more details.

The latter question is the main topic of the present text (Chaps. 3 and 4). At first one is led to restrict to those symmetric spaces, called here "special", for which the method applies in the same way as for Lie groups. But, when considering more general spaces, one needs to introduce a real-valued function $e(X, Y)$ of two tangent vectors X, Y: identically 1 for special spaces, this function embodies the modification of the Kashiwara-Vergne method required for a general symmetric space. It can be constructed from its infinitesimal structure (the corresponding Lie triple system) and contains by itself much information about invariant analysis on the symmetric space, when transferred to its tangent space at the origin via the exponential mapping (hence the notation "e"): invariant differential operators, mean values, spherical functions are related to e. The search for such relations between analysis and infinitesimal properties is our main guiding line.

Several recent works [1, 6, 12] have attracted attention on the non-unique ness in such constructions and encouraged me to rewrite my previous papers [43–47] entirely. Reorganizing them in a more synthetic way, we now separate the general use of e in invariant analysis on a symmetric space (Chap. 3) from properties arising from a specific construction of this function (Chap. 4). Not only do we speak of "an e-function" instead of "the e-function", but we also add several unpublished results, some of them recently obtained. Many proofs have been rewritten. The Kashiwara-Vergne "conjecture" is henceforth a theorem and, using it for the group G, stronger results and easier proofs can be given for the main properties of e on the symmetric space G/H (Sect. 4.4).

Let us describe the contents in more detail.

Chapter 1, devoted to the Kashiwara-Vergne method for Lie groups, provides inspiration and motivation for its extension to symmetric spaces. It is however almost entirely independent of the sequel, its results being only used in Sect. 4.4.2. We give a complete proof of the conjecture for two important families of Lie groups (quadratic, resp. solvable, following [8], resp. [48]) and a brief overview of the latest proof by Alekseev and Torossian [7] for the general case.

Apart from Chap. 1 the whole paper focuses on a **convolution transfer formula** from a symmetric space $S = G/H$ to its tangent space $\mathfrak{s}$ at the origin. Working on suitable neighborhoods of the origin, we prove the existence of a (non-unique) function e on $\mathfrak{s} \times \mathfrak{s}$ such that, for all H-invariant distributions u, v on $\mathfrak{s}$ and all test functions f on $\mathfrak{s}$,

$$\left\langle \tilde{u} *_S \tilde{v}, \tilde{f} \right\rangle = \langle u(X) \otimes v(Y), e(X,Y) f(X+Y) \rangle ,$$

where X, Y denote variables in $\mathfrak{s}$. Let us explain the notation. The function f on $\mathfrak{s}$ is transferred to S as $\tilde{f}$ by means of the exponential mapping Exp of the symmetric space and multiplication by some factor j, namely $f(X) = j(X)\tilde{f}(\operatorname{Exp} X)$. No specific choice of j is necessary up to this point, though it soon becomes clear that the most interesting example is $j = J^{1/2}$, the square root of the Jacobian of Exp. This transfer extends to distributions by duality, giving two H-invariant distributions $\tilde{u}, \tilde{v}$ on S.

The convolution product $*_S$ is defined in **Chap. 2**, where a few examples are given.

In **Chap. 3**, regardless of any construction of e, we develop the outcome of the convolution formula in H-invariant analysis on S. If e is identically 1 (the "special" case studied in Sect. 3.2), its right-hand side is simply $\langle u *_{\mathfrak{s}} v, f \rangle$, given by the classical abelian convolution on the vector space $\mathfrak{s}$. The H-invariant analysis on a special symmetric space thus boils down to classical Euclidean analysis on its tangent space at the origin. For general symmetric spaces the formula, applied to distributions supported at the origin, leads to an explicit description of G-invariant differential operators in the exponential chart and of the algebra $\mathbb{D}(S)$ of all these operators, which is commutative whenever $e(X,Y) = e(Y,X)$ (Sect. 3.3). Using e we also give, if H is compact, an expansion of mean value operators and spherical

functions (Sect. 3.4). Invariant analysis is a well-known topic if S is a Riemannian symmetric space of the noncompact type, and Sect. 3.6 is devoted to a discussion of the links between e and the classical approach in this case. Our (partly conjectural) results of this section suggest the possibility of a far-reaching generalization of Duflo's isomorphism to symmetric spaces. In Sect. 3.7 we propose an explicit e-function for isotropic Riemannian symmetric spaces, arising from manipulations of integral formulas as explained in Sect. 3.5. Two technical proofs of this chapter are postponed to the Appendix.

In **Chap. 4** we give a general construction of e for arbitrary symmetric spaces, relying on the Campbell-Hausdorff formula in the spirit of the original Kashiwara-Vergne paper. This chapter contains our main results (Theorems 4.12, 4.20, 4.22, 4.24), relating e to the infinitesimal structure of the space. Let us call the reader's attention to an element $c(X, Y)$ of the group H, constructed in Sect. 4.2.5, which plays a key role at several places (e.g. Corollary 3.17, Propositions 3.21 and 4.18, Theorem 4.24).

The theory extends to line bundles over a symmetric space. Though more general and arguably more natural (particularly for the bundle of half-densities), this framework requires handling H-components and more cumbersome notation. We made the choice to deal with line bundles in specific sections of Chaps. 2–4 only. A character χ of the subgroup H of G defines a line bundle L_χ over G/H. The convolution transfer formula still holds for H-invariant sections of L_χ provided e is replaced by e_χ, the product of e and a factor involving χ. Again the formula leads to a description in terms of e_χ of the algebra $\mathbb{D}(L_\chi)$ of invariant differential operators and a new proof of Duflo's theorem on its commutativity for certain χ, in particular for the bundle of half-densities (Sects. 3.8 and 4.5).

Sections 3.9 and 4.7 list a few open questions.

Chapters 1–4 are, to a large extent, independent of each other. The Reader is assumed to have some familiarity with the basic theory of Lie groups and symmetric spaces, as can be gleaned from the books [27, 28, 37, 58].