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Daniel Scott Farley • Ivonne Johanna Ortiz

# Algebraic K-theory of Crystallographic Groups

The Three-Dimensional Splitting Case

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ISBN 978-3-319-08152-6      ISBN 978-3-319-08153-3 (eBook)  
DOI 10.1007/978-3-319-08153-3  
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434  
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2014946579

Mathematics Subject Classification (2014): 20H15, 19B28, 19A31, 19D35

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*To the memory of Almir Alves (1965–2009)*



# Acknowledgments

The authors would like to thank Tom Farrell for originally suggesting this project to us, and for his invaluable encouragement and support while it was being completed. We are also grateful to Jean-François Lafont for his encouragement, many helpful discussions, and for his comments on preliminary versions of this manuscript. The graphics in this monograph were kindly produced by Dennis Burke.

This work is dedicated to the memory of Almir Alves. Pedro Ontaneda has written a memorial essay describing Almir's remarkable life. It can be found (at the time of this writing) at:

[www.math.binghamton.edu/dept/alves/pedro/Almir2.html](http://www.math.binghamton.edu/dept/alves/pedro/Almir2.html)

Almir grew up in poverty, attending school for the first time when he was 10 years old. He was a lifelong fighter, who flourished through perseverance, and became a Ph.D. mathematician with a gift for explaining complicated ideas in a simple way. He is greatly missed by all of us.

Pedro Ontaneda informed us that Almir, a few days before his death, told him that he knew how to prove the splitting formulas for  $K_0$  and  $K_{-1}$ .

This project was supported in part by the NSF, under the second author's grants DMS-0805605 and DMS-1207712.





# Contents

<b>1</b>	<b>Introduction</b>	1
<b>2</b>	<b>Three-Dimensional Point Groups</b>	9
2.1	Preliminaries	9
2.2	Classification of Orientation-Preserving Point Groups	11
2.3	Classification of Point Groups with Central Inversion	15
2.4	Classification of the Remaining Point Groups and Summary	15
2.5	Descriptions of Selected Point Groups	17
2.5.1	The Orientation-Preserving Standard Point Groups	17
2.5.2	The Standard Point Groups with Inversion	18
2.5.3	The Remaining Standard Point Groups	18
2.5.4	Some Non-standard Point Groups	20
<b>3</b>	<b>Arithmetic Classification of Pairs <math>(L, H)</math></b>	23
3.1	Definition of Arithmetic Equivalence and a Lemma	23
3.2	Full Sublattices in Pairs $(L, H)$ , Where $H$ Contains $(-1)$	24
3.3	Description of Possible Lattices $L$	28
3.4	Classification of Pairs $(L, H)$ , Where $(-1) \in H$	32
3.5	The Classification of the Remaining Pairs $(L, H)$	36
<b>4</b>	<b>The Split Three-Dimensional Crystallographic Groups</b>	41
<b>5</b>	<b>A Splitting Formula for Lower Algebraic <math>K</math>-Theory</b>	45
5.1	A Construction of $E_{\mathcal{FIN}}(\Gamma)$ for Crystallographic Groups	45
5.2	A Construction of $E_{\mathcal{VC}}(\Gamma)$ for Crystallographic Groups	45
5.3	A Splitting Formula for the Lower Algebraic $K$ -Theory	47
<b>6</b>	<b>Fundamental Domains for the Maximal Groups</b>	59
6.1	A Special Case of Poincare's Fundamental Polyhedron Theorem	59
6.2	Cell Structures and Stabilizers	62
6.2.1	Standard Cellulations and Equivariant Cell Structures	62
6.2.2	Computation of Cell Stabilizers and Negligible Groups	64

6.3	A Fundamental Polyhedron for $\Gamma_1$ .....	64
6.4	A Fundamental Polyhedron for $\Gamma_2$ .....	66
6.5	A Fundamental Polyhedron for $\Gamma_3$ .....	69
6.6	A Fundamental Polyhedron for $\Gamma_4$ .....	71
6.7	A Fundamental Polyhedron for $\Gamma_5$ .....	73
6.8	A Fundamental Polyhedron for $\Gamma_6$ .....	75
6.9	A Fundamental Polyhedron for $\Gamma_7$ .....	77
<b>7</b>	<b>The Homology Groups <math>H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty})</math></b> .....	<b>81</b>
7.1	The Algebraic $K$ -Theory of Cell Stabilizers in $E_{\mathcal{FIN}}(\Gamma)$ .....	82
7.1.1	The Lower Algebraic $K$ -Theory of $\mathbb{Z}/4 \times \mathbb{Z}/2$ .....	83
7.1.2	The Lower Algebraic $K$ -Theory of $\mathbb{Z}/6 \times \mathbb{Z}/2$ .....	84
7.1.3	The Lower Algebraic $K$ -Theory of $A_4 \times \mathbb{Z}/2$ .....	87
7.2	The Homology of $E_{\mathcal{FIN}}(\Gamma)$ .....	88
7.3	Calculations of $H_n^{\Gamma}(E_{\mathcal{FIN}}(\Gamma); \mathbb{K}\mathbb{Z}^{-\infty})$ .....	90
<b>8</b>	<b>Fundamental Domains for Actions on Spaces of Planes</b> .....	<b>99</b>
8.1	Negligible Line Stabilizer Groups .....	99
8.2	The Finiteness of the Indexing Set $\mathcal{T}''$ .....	102
<b>9</b>	<b>Cokernels of the Relative Assembly Maps for <math>\mathcal{VC}_{\infty}</math></b> .....	<b>119</b>
9.1	Passing to Subgroups .....	119
9.2	Reconstructing $\Gamma_{\ell}$ from $\overline{\Gamma}_{\ell}$ .....	124
9.3	Cokernels of Relative Assembly Maps .....	131
9.3.1	The Lower Algebraic $K$ -Theory of $C_4 \times \mathbb{Z}$ , $D_4 \times \mathbb{Z}$ , and $D_6 \times \mathbb{Z}$ .....	131
9.3.2	The Lower Algebraic $K$ -Theory of $D_2 \rtimes_{\alpha} \mathbb{Z}$ .....	134
9.3.3	The Lower Algebraic $K$ -Theory of $D_4 *_C C_4$ .....	135
9.3.4	The Lower Algebraic $K$ -Theory of $C_4 \times D_{\infty}$ .....	135
9.3.5	The Lower Algebraic $K$ -Theory of $D_6 \times D_{\infty}$ .....	136
<b>10</b>	<b>Summary</b> .....	<b>137</b>
	<b>References</b> .....	<b>143</b>
	<b>Index</b> .....	<b>147</b>