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Finiteness Properties of Arithmetic Groups Acting on Twin Buildings

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Münster, Germany Stefan Witzel

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Introduction

In these notes we determine finiteness properties of two classes of groups whose most prominent representatives are the groups

$$\mathrm{SL}_n(\mathbb{F}_q[t]) \qquad \text{and} \qquad \mathrm{SL}_n(\mathbb{F}_q[t,t^{-1}]),$$

the special linear groups over the ring of polynomials, respectively, Laurent polynomials, over a field with q elements.

The finiteness properties we are interested in generalize the notions of being finitely generated and of being finitely presented. A group G is generated by a subset S if and only if the Cayley graph $\mathrm{Cay}(G,S)$ is connected. And S is finite if and only if the quotient $G \backslash \mathrm{Cay}(G,S)$ is compact. That is, G is finitely generated if and only if it admits a connected Cayley graph that has compact quotient modulo G.

Similarly, consider a set R of relations in G, that is, words in the letters $S \cup S^{-1}$ which describe the neutral element in G. The Cayley 2-complex $\mathrm{Cay}(G,S,R)$ is obtained from the Cayley graph by gluing in a 2-cell for every edge loop that is labeled by an element of R. The Cayley 2-complex is 1-connected (that is, connected and simply connected) if and only if $\langle S \mid R\rangle$ is a presentation of G. And it has a compact quotient modulo G if both S and R are finite. That is, G is finitely presented, if and only if G admits a 1-connected, cocompact Cayley 2-complex.

Since G is described up to isomorphism by a presentation, this is how far the classical interest goes. But from the topological point of view one can go on and ask whether it is possible to glue in 3-cells along "identities" I in such a way that the resulting complex $\mathrm{Cay}(G,S,R,I)$ is 2-connected and cocompact.

Wall [Wal65, Wal66] developed this topological point of view and introduced the following notion: a group G is of type F_n if it acts freely and cocompactly on a contractible CW-complex X such that the quotient $G\backslash X^{(n)}$ of the n-skeleton modulo G is compact. It is not hard to see that, indeed, a group is of type F_1 if and only if it is finitely generated, and is of type F_2 if and only if it is finitely presented. We say that a group is of type F_∞ if it is of type F_n for all n. This property is strictly weaker than that of being of type F, namely having a cocompact classifying space. In fact, a group that has torsion elements cannot be of type F. But if it is virtually

of type F, that is, if it contains a finite index subgroup that is of type F, then it is still of type F_∞.

In the decades following Wall's articles some effort has been put, on the one hand, in determining what finiteness properties certain interesting groups have, and on the other hand, in better understanding what the properties F_n mean by producing separating examples. We mention just some of the results not directly related to the present notes. Statements about arithmetic and related groups will be mentioned further below. It will be convenient to introduce the finiteness length of a group G defined as

$$\phi(G) := \sup\{n \in \mathbb{N} \mid G \text{ is of type } F_n\}.$$

Finitely generated groups that are not finitely presented have been known since Neumann's article [Neu37]. The first group known to be of type F_2 but not of type F_3 was constructed by Stallings [Sta63]. Stallings's example is the case $n = 3$ of the following construction due to Bieri: let L^n be the direct product of n free groups on two generators and let K_n be the kernel of the homomorphism $L^n \to \mathbb{Z}$ that maps each of the canonical generators to 1. In [Bie76] Bieri showed that $\phi(K_n) = n - 1$. Abels and Brown [AB87, Bro87] proved that the groups $\mathbf{G}_n$ of upper triangular n-by-n matrices with extremal diagonal entries equal to 1 satisfy $\phi(\mathbf{G}_n(\mathbb{Z}[\frac{1}{p}])) = n-1$ for any prime p. Brown [Bro87] also proved that Thompson's groups and some of their generalizations are of type F_∞. For the group $\mathbf{B}_n$ of upper triangular matrices and a ring $\mathcal{O}_S$ of S-integers of a global function field (defined below), Bux [Bux04] showed that $\phi(\mathbf{B}_n(\mathcal{O}_S)) = |S| - 1$.

The general pattern of proof to determine the finiteness properties of a group G is the same in many cases: first one produces a contractible CW-complex X on which G acts with "good" (often finite) stabilizers. This action will typically not be cocompact. One then constructs a filtration $(X_i)_i$ of X by cocompact subcomplexes X_i such that the inclusions $X_i \hookrightarrow X_j$, $i \leq j$, preserve $(n-1)$-connectedness for some fixed n. Now the n-skeleton of X_0 is the n-skeleton of a contractible space on whose n-skeleton G acts cocompactly. This would show that G is of type F_n if the stabilizers were trivial rather than just "good." In this situation there is a famous criterion due to Brown [Bro87] stating not only that "good" stabilizers are good enough to conclude that G is of type F_n, but also that the group is not of type F_{n+1} provided the filtration does not preserve n-connectedness in an essential way.

In some cases an appropriate space X for G to act on has been known long before people were interested in higher finiteness properties. Thus Raghunathan [Rag68] showed that arithmetic subgroups of semisimple algebraic groups over number fields, like $\mathrm{SL}_n(\mathbb{Z})$, are virtually of type F. To this end he considered the action of the arithmetic group on the symmetric space X of its ambient Lie group and constructed a Morse function on the quotient $G\backslash X$ with compact sublevel sets. It is noteworthy that this proof fits into the general pattern described above. In fact, the filtrations mentioned before are often, and in these notes in particular, obtained by (a discrete version of) Morse theory. This reduces the problem to understanding certain local data, the descending links.

There are two classes of groups that are closely related to arithmetic groups: mapping class groups $\mathrm{Mod}(S_g)$ of closed surfaces and outer automorphism groups $\mathrm{Out}(F_n)$ of finitely generated free groups. The space for $\mathrm{Mod}(S_g)$ to act on is Teichmüller space, likewise a very classical object. A proof that Teichmüller space admits an invariant contractible cocompact subspace, and therefore $\mathrm{Mod}(S_g)$ is virtually of type F, can be found in [Iva91]. The right space to consider for $\mathrm{Out}(F_n)$ is outer space [VC86]. Unlike the previous classical spaces, outer space was not known before Culler and Vogtman constructed it to establish that $\mathrm{Out}(F_n)$ is virtually of type F. The cited proofs for mapping class groups and outer automorphism groups of free groups are very similar in spirit to the one for arithmetic groups and fit again into our general pattern. An alternative to exhibiting a highly connected cocompact subspace of the original space is to construct a cocompact partial compactification on which the group still acts properly discontinuously. This has been done by Borel and Serre [BS73] for arithmetic groups and by Harvey [Har79] for mapping class groups.

A number theoretic generalization of arithmetic groups is S-arithmetic groups. To define them, we consider a number field k and its set of places T, that is, a maximal set of inequivalent valuations. Let T_∞ denote the subset of Archimedean places, such as the usual absolute value. For an element $\alpha \in k$ the condition that $v(\alpha) \leq 1$ for all non-Archimedean places v describes the ring of integers of k. If instead one imposes this condition for all but a finite set S of non-Archimedean places, one obtains the ring of S-integers $\mathcal{O}_S$. Accordingly, S-arithmetic groups are matrix groups of S-integers.

The field k admits a completion k_v with respect to every valuation $v \in T$. An S-arithmetic group $\mathbf{G}(\mathcal{O}_S)$ is a discrete subgroup of the locally compact group $\prod_{v \in T_\infty \cup S} \mathbf{G}(k_v)$. For instance, the group $\mathrm{SL}_n(\mathbb{Z}[\frac{1}{2}])$ is a discrete subgroup of the group $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_2)$.

If $\mathbf{G}$ is a reductive k-group, then $\mathbf{G}(\mathcal{O}_S)$ acts properly discontinuously on the product of the spaces X_v associated with the locally compact groups $\mathbf{G}(k_v)$, $v \in T_\infty \cup S$. For the Archimedean places, this is again a symmetric space. For the non-Archimedean places, the naturally associated space is a Bruhat–Tits building, which is a locally compact cell complex with a piecewise Euclidean metric.

The action of an S-arithmetic subgroup of a reductive algebraic group over a number field described above has been used by Borel and Serre [BS76, Théorème 6.2] to show that these groups are virtually of type F.

There is the notion of a global function field which parallels that of a number field. A global function field k is a finite extension of a field of the form $\mathbb{F}_p(t)$ where $\mathbb{F}_p$ is the finite field with p elements and t is transcendental over $\mathbb{F}_p$. Global function fields resemble number fields in the valuations that they admit. In particular places and S-integers can be defined analogously, with the exception that there are no Archimedean places. As in the number field case, if $\mathbf{G}$ is a reductive group over k, then $\mathbf{G}(\mathcal{O}_S)$ is naturally a discrete subgroup of $\prod_{v \in S} \mathbf{G}(k_v)$ and therefore acts on the associated space $X = \prod_{v \in S} X_v$, which is a building since there are no Archimedean places. The dimension $d(\mathbf{G}, S) := \dim X$ can be described algebraically as the sum of the ranks of $\mathbf{G}$ over the local fields k_v.

Finiteness properties of S-arithmetic subgroups of reductive groups over global function fields differ fundamentally from the analogous properties in the number field case that we have seen above (always F_∞). This is apparent already from the smallest example: Nagao [Nag59] showed that the groups $\mathrm{SL}_2(\mathbb{F}_q[t])$ are not finitely generated. Over the years, various mathematicians investigated other S-arithmetic subgroups of reductive groups. In fact, it suffices to study subgroups of almost simple groups. Behr started by determining which of the groups are finitely generated [Beh69] and which are finitely presented [Beh98].

Concerning higher finiteness properties, Stuhler [Stu80] concentrated on the group SL_2 and showed that $\mathrm{SL}_2(\mathcal{O}_S)$ has finiteness length $|S| - 1$. In a different direction, Abels and Abramenko [Abr87, Abe91, AA93] concentrated on the rings $\mathcal{O}_S = \mathbb{F}_q[t]$ (where S contains only one place) and showed that the groups $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ have finiteness length $n - 1$ provided q is large enough. This was later extended by Abramenko [Abr96] to groups $\mathbf{G}(\mathbb{F}_q[t])$ where $\mathbf{G}$ is a classical group.

All of the above results show in their specific situation that the finiteness length is

$$\phi(\mathbf{G}(\mathcal{O}_S)) = d(\mathbf{G}, S) - 1. \qquad (*)$$

This caused Brown [Bro89, p. 197] (and possibly others before him) to ask whether this would always be the case. As evidence got stronger, the assertion that $(*)$ holds for any almost simple group $\mathbf{G}$ (with some obvious exceptions) became known as the Rank Conjecture.

That the finiteness properties cannot be better than predicted, i.e. the inequality $\phi(\mathbf{G}(\mathcal{O}_S) \leq d(\mathbf{G}, S) - 1$, was proven by Bux and Wortman [BW07]. An alternative proof of this fact that applies to a more general situation has been given by Gandini [Gan12] using work of Kropholler [Kro93, KM98]. Concerning negative statements about finiteness properties, the most recent result is by Wortman [Wor13] who showed that $\mathbf{G}(\mathcal{O}_S)$ has a finite-index subgroup Γ with $H_d(\Gamma, \mathbb{F}_p)$ infinite (p the characteristic of k).

Continuing with positive results, Bux and Wortman [BW11] showed that $(*)$ holds provided $\mathbf{G}$ has rank one over the field k. Finally the conjecture became the Rank Theorem by joint work [BKW13] of Bux, Köhl and the author:

Rank Theorem. *Let k be a global function field. Let $\mathbf{G}$ be a connected, non-commutative, absolutely almost simple k-isotropic k-group. Let $d := \sum_{s \in S} \mathrm{rank}_{k_s} \mathbf{G}$ be the sum over the local ranks at places $s \in S$ of $\mathbf{G}$. Then $\mathbf{G}(\mathcal{O}_S)$ is of type F_{d-1} but not of type F_d.*

At this point some words about the proof are in order. We know already that $\mathbf{G}(\mathcal{O}_S)$ acts on a building X with finite stabilizers. The strategy is of course to produce a cocompact filtration $(X_i)_i$ of X and to investigate the relative connectivity of the X_i. To produce such a filtration, Harder's reduction theory [Har67, Har68, Har69] is used. This is a deep and powerful theory which describes (an invariant family of) horoballs that can be removed from the building to obtain a

cocompact space. However it is relatively difficult to analyze the connectivity of the subspaces determined by Harder's reduction theory.

For that reason, the above partial positive results can be divided into two classes, according to how they deal with this difficulty. Stuhler's result [Stu80] and the result by Bux and Wortman [BW11] restrict to the case where $\mathbf{G}$ has global rank one. This allows one to choose the horoballs disjointly which makes analyzing the connectivity a little easier.

On the other hand, Abels and Abramenko [Abr87, Abe91, AA93, Abr96] restrict the ring $\mathcal{O}_S$ to be $\mathbb{F}_q[t]$ (or $\mathbb{F}_q[t,t^{-1}]$ as far as the general method is concerned). In this situation, Harder's reduction theory can be replaced by the theory of twin buildings which is much more explicit. In these notes we will follow the second strategy and prove those cases of the Rank Theorem that can be treated using twin buildings instead of reduction theory. Our goal is to prove:

Main Theorem. *Let* $\mathbf{G}$ *be a connected, non-commutative, absolutely almost simple* $\mathbb{F}_q$*-group of* $\mathbb{F}_q$*-rank* $n \geq 1$. *Then* $\mathbf{G}(\mathbb{F}_q[t])$ *is of type* F_{n-1} *but not of type* F_n *and* $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ *is of type* F_{2n-1} *but not of type* F_{2n}.

The first part of the Main Theorem is proved in Chap. 2 as Theorem 2.73. The second part is proved in Chap. 3 as Theorem 3.35.

The general setup is the same as in the Rank Theorem. The rings $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t,t^{-1}]$ are rings of S-integers in $\mathbb{F}_q(t)$ where $S = \{v_\infty\}$ contains one place in the first case and $S = \{v_0, v_\infty\}$ contains two places in the second case. So the groups are S-arithmetic groups and in particular are discrete subgroups of locally compact groups $\mathbf{G}(\mathbb{F}_q[t]) \subseteq \mathbf{G}(\mathbb{F}_q(\!(t^{-1})\!))$ and $\mathbf{G}(\mathbb{F}_q[t,t^{-1}]) \subseteq \mathbf{G}(\mathbb{F}_q(\!(t^{-1})\!)) \times \mathbf{G}(\mathbb{F}_q(\!(t)\!))$. Since $\mathbf{G}$ is almost simple, there are irreducible Bruhat–Tits buildings X_∞ and X_0 associated to $\mathbf{G}(\mathbb{F}_q(\!(t^{-1})\!))$ and $\mathbf{G}(\mathbb{F}_q(\!(t)\!))$. The group $\mathbf{G}(\mathbb{F}_q[t])$ acts properly discontinuously on X_∞ and $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ acts properly discontinuously on $X_\infty \times X_0$. The action is not cocompact and we want to construct a cocompact filtration which preserves high connectivity.

What is special in the situation of the Main Theorem is that the group $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ happens to also be a Kac–Moody group. In terms of spaces this means that the two buildings X_0 and X_∞ that the group acts on form a twin building. That is, there is a codistance between X_0 and X_∞ measuring in some sense the distance between cells in the two buildings, and this codistance is preserved by $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$. In fact one can define two kinds of codistance: one is a combinatorial codistance between the cells of X_0 and of X_∞ and the other is a metric codistance between the points of X_0 and of X_∞. The group $\mathbf{G}(\mathbb{F}_q[t])$ is the stabilizer in $\mathbf{G}(\mathbb{F}_q[t,t^{-1}])$ of a cell in X_0.

In [Abr96] Abramenko used the combinatorial codistance to define a Morse function on X_∞ and partially obtain the first case of the Main Theorem as described above. To ensure that the filtration preserves connectedness properties, Abramenko had to study certain combinatorially described subcomplexes of spherical buildings, which arose as descending links.

In our proof we use the metric codistance in a similar way to Abramenko's use of the combinatorial codistance. The descending links that occur in our filtration

are metrically described subcomplexes of spherical buildings. The connectivity properties of these have already been established by Schulz [Sch13].

Since our proof makes heavy use of the piecewise Euclidean metric on the buildings X_0 and X_∞ it is restricted to affine Kac–Moody groups. Abramenko's combinatorial proof, on the other hand, making no reference to the metric structure of the twin building, generalizes to hyperbolic Kac–Moody groups.

In that sense we profit from working in the intersection of two worlds: that of S-arithmetic groups and that of Kac–Moody groups. On the other hand, it is fair to say that after proving the Main Theorem all that is additionally needed to prove the Rank Theorem is related to reduction theory or the theory of algebraic groups (which is not little of course). For that reason understanding the proof of the Main Theorem is a good way to assemble the tools for the Rank Theorem without having to deal with reduction theory. Among those tools is the flattening of level sets that is introduced in Sects. 2.4 and 2.5. Another technique is the use of the depth function as a secondary height function in the flattened regions. It was introduced in [BW11] and is generalized to reducible buildings in Sect. 2.7.

In Appendix A we show that the finiteness length of an almost simple S-arithmetic group can only grow as S gets larger (a fact that was already used in [Abr96]). Though this is clear in the presence of the Rank Theorem, it allows one to deduce finiteness properties (though not the full finiteness length) of some groups even without it. For example, the following is a consequence of our Main Theorem:

Corollary. *Let* $\mathbf{G}$ *be a connected, non-commutative, absolutely almost simple* $\mathbb{F}_q$*-group of* $\mathbb{F}_q$*-rank* $n \geq 1$*. Let* S *be a finite set of places of* $\mathbb{F}_q(t)$ *and let* $G := \mathbf{G}(\mathcal{O}_S)$*. If* S *contains* v_0 *or* v_∞*, then* G *is of type* F_{n-1}*. If* S *contains* v_0 *and* v_∞*, then* G *is of type* F_{2n-1}*.*

These notes are based on the author's Ph.D. thesis [Wit11].