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A Panorama of Discrepancy Theory

Springer

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Preface

Discrepancy theory concerns the problem of replacing a continuous object with a discrete sampling. The roots of the theory are in H. Weyl's fundamental work on uniformly distributed sequences [55]. *Just sequences* have been sought for a long time and in 1935, J.G. van der Corput conjectured their non-existence:

> If $s_1, s_2, s_3, \ldots$ is an infinite sequence of real numbers lying between 0 and 1, then corresponding to any arbitrarily large k, there exist a positive integer n and two subintervals, of equal length, of $(0, 1)$, such that the number of s_ν $(\nu = 1, \ldots, n)$ that lie in one of the subintervals differs from the numbers of such s_ν that lie in the other subinterval by more than k.

The conjecture was proved in 1945 by T. van Aardenne-Ehrenfest [53] and then put in a quantitative form by her in 1949 [54]. In 1954, her quantitative result was improved by K.F. Roth [46], who observed that the study of the discrepancy of an infinite sequence in the unit torus $\mathbb{T}$ is equivalent to the study of the discrepancy of a finite set in $\mathbb{T}^2$, thereby introducing a geometric point of view in the study of *irregularities of point distribution*, or *discrepancy theory*. For many years, the main results were obtained by Roth and W.M. Schmidt. However, over the last 30 years, many mathematicians and computer scientists have developed the original theory and explored new directions and applications.

Discrepancy theory is currently at a crossroads between number theory, combinatorics, Fourier analysis, algorithms and complexity, probability theory, and numerical analysis. Its current applications range from traditional science and engineering to modern computer science and financial mathematics.

The theoretical aspects of the theory are mathematical in nature and cover two major areas of research:

1. Classical and geometric discrepancy theory (e.g., low discrepancy sequences, geometric discrepancy, number theoretical aspects)
2. Combinatorial discrepancy theory (e.g., hypergraph coloring, arithmetic structures, discrepancy games)

The applied and constructive aspects of the theory are mathematical and computational in nature and include the following two major areas of research:

3. a. Algorithms and complexity (e.g., complexity of construction of point sets, pseudorandomness, derandomization, data structures in computational geometry)
   b. Quasi Monte Carlo methods and numerical integration (e.g., high dimensional integration, complexity of integration, computational finance)

There are several excellent books on discrepancy theory (Beck and Chen [7]; Drmota and Tichy [22]; Matoušek [32], [33, new edition 2010]; and Chazelle [14]) and the book chapter of Beck and Sós in the *Handbook of Combinatorics* [10], but perhaps no one of them actually showed the present variety of points of view and applications covering the areas 1–3. In the last 15 years, several conferences on discrepancy theory addressed the integrated view on discrepancy theory, like *Discrepancy Theory and its Application*, Kiel 1998 (org.: Matoušek, Srivastav), Oberwolfach 2004 (org.: Chazelle, Chen, Srivastav), and *Discrepancy Theory and Related Areas*, Varenna 2007 (org.: Chen, Srivastav, Travaglini). In these meetings a vivid exchange of ideas and concepts between different fields in discrepancy theory was visible, as well as the need for an integrated presentation of discrepancy theory and its applications, which might be helpful to bridge the various areas, with focus on the state of the art. Such a book should also be an invitation to researchers and students to find a quick way into the different methods and to motivate interdisciplinary research.

For the above reasons, we started planning a book consisting of several chapters, written by experts in the specific fields, and focused on different aspects of the theory. The chapters are not traditional expository papers, but rather detailed introductions to different areas of discrepancy theory, and should provide an effective and reasonably self-contained basis for study and further research. Let us briefly describe the new contribution of the book to the topics 1–3.

1. Classical and Geometric Discrepancy Theory: The irregularity of the distribution of a finite point set in d-dimensional Euclidean space with respect to a collection of geometric structures in the unit cube is a classical problem. It has two major aspects. First of all, there are lower bounds which confirm that such irregularity must exist for every point set. Of equal importance are upper bounds which restrict the size of such irregularities by good placements of the points of the point set. Here the collection of geometric structures may concern rectangles, squares, polygons, balls, convex regions, with contraction, translation and/or rotation, and the size of the irregularity depends on the choice of the collection of geometric structures involved. A classical result here is the full determination of the size of the mean square discrepancy with respect to the collection of aligned rectangular boxes by Roth [46, 48], together with the breakthrough by Chen and Skriganov [16] on explicit constructions. New insights have also been presented by the number theoretic work of Skriganov (1994–2001) involving point sets obtained from lattices over totally real number fields.

Lower bound techniques, such as the orthogonal function method of Roth, the Fourier transform technique of Beck, and the integral geometric method of Alexander, have found applications in the study on algorithms and complexity. Upper bound considerations have found applications in quasi Monte Carlo methods and numerical integration. Apart from the pioneering work of Roth, there have been many notable successes, particularly the work of Alexander and Beck on lower bounds and the work of Beck, Chen, and Skriganov on upper bounds.

Four chapters in the book will address these aspects and new developments.

In Chap. 1, William Chen and Maxim Skriganov provide a detailed study of upper bounds, using arguments from diophantine approximation, probability theory, number theory, and Fourier analysis. The chapter is also an introduction to basic concepts and proofs, like probabilistic and deterministic techniques and their comparison, van der Corput sets, Fourier–Walsh analysis, explicit constructions, and orthogonality.

In Chap. 2, Dmitriy Bilyk will present the recent breakthrough by Bilyk, Lacey, and Vagharshakyan on the L^∞ discrepancy problem related to rectangles in dimension $d \geq 3$. This was called the *Great open problem* in the first chapter of the book of Beck and Chen [7]. Until recently, there was only one deep result by Beck [6] in dimension 3. Bilyk will bring the reader to the core of the problem and will show the connections of discrepancy to other areas of mathematics, in particular to the *small ball inequality* which arises in harmonic analysis as well as in the study of the small deviation probabilities for the Brownian sheet.

As we said, a famous problem in the classical geometric theory concerns the study of discrepancies related to discs and more generally to convex bodies; see Schmidt [49], Beck [4, 5], Beck and Chen [8], and Montgomery [37]. Most of these results depend on suitable estimates for the average decay of certain Fourier transforms, a topic which has been recently investigated by a number of authors, among them Brandolini, Colzani, and Travaglini [11], and Brandolini, Hofmann, and Iosevich [12]. In Chap. 3, Luca Brandolini, Giacomo Gigante, and Giancarlo Travaglini will present in a detailed and unified way recent Fourier analysis results and their connections to the above discrepancy problem.

Finding the integer solutions of a Pell equation is equivalent to finding the integer lattice points in a long and narrow tilted hyperbolic region, where the slope is a quadratic irrational. Motivated by this relationship, in Chap. 4, József Beck presents a systematic study of point counting with respect to translated or congruent families of any given long and narrow hyperbolic region. The main results exhibit a fascinating new phenomenon about the extra large discrepancy called *superirregularity* and demonstrate, in a quantitative sense, that in point counting with respect to translated/congruent copies of any long and narrow hyperbolic region, superirregularity is inevitable. The techniques involved depend on ideas from number theory, combinatorics, probability theory, and Fourier analysis.

2. Combinatorial Discrepancy Theory: The basic problem in combinatorial discrepancy theory is to color the nodes of a finite hypergraph with two colors in a way that ideally in every hyperedge the number of nodes in the two colors is the

same. The minimum deviation from this optimal situation is called the discrepancy of the hypergraph.

There is a relation between combinatorial discrepancy and classical geometric discrepancy, usually known as the transfer lemma, which allows the transfer of results between geometric discrepancy theory and combinatorial discrepancy theory. On the other hand, for many important problems in combinatorial discrepancy theory the transfer lemma is too weak, and intrinsic combinatorial methods are required.

The foundation of combinatorial discrepancy theory was laid by the work of Beck [3], Beck and Fiala [9], Spencer [52] (*the six-standard-deviation theorem*), and Lovász, Spencer, and Vesztergombi [30]. Combinatorial discrepancies arise in several areas of combinatorics, like Ramsey theory, uni-modular matrices, and extremal set systems. Over the last 10 years a number of new results have appeared, leading to new techniques and sometimes to optimal discrepancy bounds. Among them are the trace bound for the hereditary discrepancy of Chazelle and Lvov [15], the new bounds for geometric set systems of Matoušek, Welzl, and Wernisch [36], Matoušek [31], Chazelle [13], and the resolution of the linear discrepancy conjecture for totally unimodular matrices by Doerr [19]. A new aspect has been the investigation of multicolor discrepancy by Doerr and Srivastav [20], where some unexpected phenomena arise by passing from two colors to several colors.

Among the interesting classes of hypergraphs are certainly those with some arithmetic structures, like the hypergraph of arithmetic progressions in the first n integers (Roth [47]; Matoušek and Spencer [35]) and their generalizations, like products and sums of arithmetic progressions (Doerr, Srivastav and Wehr [21]; Hebbinghaus [27]; Přívětivý [45]) or hyperplanes in finite vector spaces (Hebbinghaus, Schoen and Srivastav [28]).

Quite recently, an efficient randomized algorithm for the construction of a 2-coloring satisfying Spencer's famous six-standard-bound was given by Bansal [1], and in a derandomized version by Bansal and Spencer [2], resolving a long-standing open problem.

In this book, three chapters are concerned with the development of combinatorial discrepancy theory and some of the mentioned directions.

In Chap. 5 on multicolor discrepancy of arithmetic structures, Nils Hebbinghaus and Anand Srivastav present the discrepancy theory for hypergraphs with arithmetic structures, e.g., arithmetic progressions in the first N integer, their various generalizations, like cartesian products, sums of arithmetic progressions, central arithmetic progressions (Bohr sets) in $\mathbb{Z}_p$, and linear hyperplanes in finite vector spaces. At the beginning, the theory of multicolor discrepancy is described, like upper and lower bounds for general hypergraphs and the multicolor generalization of several classical 2-color theorems. It is shown that at several places phenomena not visible in the 2-color theory show up, among them the multicoloring of products of hypergraphs. The focus of the chapter are proofs of lower bounds for the multicolor discrepancy for the hypergraphs mentioned above, where often the application of Fourier analysis or linear algebra techniques is not sufficient and has to be combined with combinatorial arguments.

Chapter 6 by Nikhil Bansal comprises recent breakthrough work on algorithms in combinatorial discrepancy theory. Since 1985 it has been an open problem, whether there is a polynomial-time algorithm which computes for a hypergraph with n nodes and n hyperedges a 2-coloring satisfying Joel Spencer's famous six-standard-deviation bound of $O(\sqrt{n})$ published in the *Transaction of the American Mathematical Society* in 1985. In 2010 N. Bansal [1] solved this problem, with a randomized algorithm based on semi-definite programming and a kind of twofold randomization. In 2011 Bansal and Spencer [2] were able to derandomize the algorithm. Some other exciting developments related to Bansal's work, for example the linear algebra technique of Lovett and Meka [50] and the tightness of the determinant bound for hereditary discrepancy due to Matoušek [34], are discussed as well.

Combinatorics is more and more touched by computational advances in computer science, and practical and efficient algorithms are sought. This is in particular true for discrepancy theory, where we wish to construct point sets or colorings satisfying the best known discrepancy bounds or finding experimental evidence for discrepancy bounds, e.g., the unsolved conjecture of Paul Erdős on the discrepancy of arithmetic progressions in the integers which is part of the *polymath project* initiated by Timothy Gowers in 2009. In Chap. 7, Lasse Kliemann shows how to efficiently compute low-discrepancy colorings using high-performance computing. As a benchmark problem he chooses the hypergraph of arithmetic progressions in the first N integers, for which the optimal discrepancy is $\Theta(N^{1/4})$ up to constants; Roth [47], Matoušek, Spencer [35]. With Bansal's algorithm one can compute, in randomized polynomial time, a coloring with discrepancy $O(N^{1/4}(\log N)^k)$ using semidefinite programs. But as the semidefinite programs grow in the number of hyperedges, the time complexity is too high even for moderately large N. Kliemann devised a new evolutionary algorithm based on estimation of distribution (EDA) on modern multicore computers. The algorithms compute the optimum $\Theta(N^{1/4})$ up to a constant factor for at least up to $N = 250{,}000$, where we have the astronomical number of $377 \cdot 10^9$ arithmetic progressions.

3. Applications and Constructions: A fundamental problem is the efficient construction of point sets with low geometric discrepancy. Classical constructions are the point sets of Halton [26], Faure [23], and Niederreiter [38]. New constructions are based on the so-called rank-1 lattice rules by Sloan, Kuo and Joe [51], Kuo [29], Dick [17], and Nuyens and Cools [42, 43]. A comprehensive theory of general tractability was developed by Gnewuch and Woźniakowski [24, 25]. The books of Novak and Woźniakowski [39–41] summarize the current state of the art in tractability theory.

Among the prominent applications of discrepancy theory are counting problems in number theory, for example the investigation of the distribution of the solutions of a diophantine equation, or the numerical integration in d-dimensional space, where d is large (e.g., between 30 and 360 in financial mathematics), the so-called quasi Monte Carlo method.

In many experiments the quasi Monte Carlo method is superior to the widely applied Monte Carlo simulation, where random points are used (see Owen [44]). However, the dispute for which applications this observation holds is ongoing

and an interesting area of research (see Niederreiter [38], Matoušek [32], Dick and Pillichshammer [18]). On the other hand, a flow of results, particularly by Woźniakowski (1998–2001), determining the complexity of approximation of numerical integration, have shown the worst case limits of approximation. Other results can be found in recent volumes of the *Journal of Complexity*.

In Chap. 8, Ákos Magyar studies the distribution of the solutions of a diophantine equation when projected onto the unit level surface via the dilations, and also when mapped to the flat torus $\mathbf{T}^n$. He obtains quantitative estimates on the rate of equi-distribution in terms of upper bounds on the associated discrepancy. The main technical tool is the Hardy–Littlewood method of exponential sums utilized to obtain asymptotic expansions of the Fourier transform of the solution sets.

Chapter 9 by Josef Dick and Friedrich Pillichshammer is devoted to the application of discrepancy theory to quasi Monte Carlo integration, with an emphasis on explicit error bounds. The chapter is a presentation of the state-of-the-art methods for quasi Monte Carlo integration.

Last, but not least, we return to the basic problem of the geometric discrepancy, where the known bounds and methods have been described in the first chapter. Now we are concerned with efficient constructions of point sets and computation of the discrepancy. In Chap. 10, Carola Doerr, Michael Gnewuch, and Magnus Wahlström present randomized and de-randomized algorithms for the construction of low discrepancy point sets and the calculation of the star discrepancy, prove complexity results, and show interesting and promising connections to integer programming.

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