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Brownian Motion and its Applications to Mathematical Analysis

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To Janina Burdzy, my mother, who taught me the foundations of probability

Preface

Depending on whom you ask, Brownian motion was either discovered or invented by Louis Bachelier in 1900 (according to economists), by Albert Einstein in 1905 (according to physicists), or by Norbert Wiener in 1923 (according to mathematicians). Biologists do not seem to be interested much in taking credit for the discovery; perhaps, they consider it obvious that they get the credit because Brownian motion is named after a biologist, Robert Brown. He observed random motion of pollen grains looking through a microscope in 1827. According to Wikipedia [Wik12a], the history of Brownian motion can be traced to ancient Rome where Lucretius, in 60 BC, wrote a remarkable description of Brownian motion of dust particles and used this as a proof of the existence of atoms. The same article in Wikipedia claims that Brownian motion was "discovered" multiple times and earlier than by people normally given credit for the discovery.

Brownian motion is a good model for a wide range of real random phenomena, from chaotic oscillations of microscopic objects, such as flower pollen in water, to stock market fluctuations. Brownian motion is also a purely abstract mathematical tool which can be used to prove theorems in "deterministic" fields of mathematics. These lecture notes contain an introduction to the applications of Brownian motion to analysis.

I do not think that there is a well-defined area of probability called "applications of Brownian motion to analysis" and I will not try to create such a field in these notes. Instead, I will present a number of diverse applications of Brownian motion to analysis and, more generally, connections between Brownian motion and analysis. All I can hope for is that summer school participants and other readers will be infected by my enthusiasm and will try to find their own research projects in this area.

Probability theory started as a science describing real-life random phenomena and only more recently developed into a branch of mathematics that generates tools which can be used to prove results in other branches of mathematics. Perhaps the best known application of probability to proving deterministic mathematical theorems is known as the "Erdös probabilistic method." According to Wikipedia [Wik12b], "it is a nonconstructive method, primarily used in combinatorics, for

proving the existence of a prescribed kind of mathematical object. It works by showing that if one randomly chooses objects from a specified class, the probability that the result is of the prescribed kind is more than zero." Another Wikipedia article [Wik12c] lists a number of applications of probability to proving theorems in the following fields of mathematics: analysis, combinatorics, algebra, topology, geometry, number theory, and quantum theory.

The history of applications of Brownian motion to analysis goes back at least to several papers of Kakutani in the mid-1940s (see [Kak44, Kak45a, Kak45b]). Other early players in the field were Burkholder [Bur76], Davis [Dav75, Dav79a], and Doob [Doo61]. I apologize to all my colleagues who contributed to the field but are not mentioned in my Twitter-like history of the subject. I have unfairly listed only some mathematicians who are independently famous. Since 1980, interactions between Brownian motion and stochastic analysis, on the one hand, and various branches of analysis, on the other hand, have become so diverse and numerous that I do not feel competent to provide an even remotely accurate review.

The topics for these notes were selected either because I did research on them or because I considered them elegant. Chapter 1 contains a very general review of Brownian motion. Chapter 2 is concerned with probabilistic proofs of classical theorems in analysis. All the remaining chapters contain mathematics developed after 1990.

I regret to say that these notes do not present material at the level of rigor that is expected from journal articles or textbooks published in mathematics. It would take several hundred pages to present rigorously all the mathematical tools needed in these notes. I kindly request that the reader consult one or more of several excellent books that provide an introduction to Brownian motion and its relationship to analysis. Personally, I often consult books by Bass [Bas95], Karatzas and Shreve [KS91], Knight [Kni81], Mörters and Peres [MP10], and Revuz and Yor [RY99]. Other books are referred to in the body of the text.

These notes contain previously published material, in the sense of mathematical substance. To large extent, they are a concatenation of many articles, edited to various degree. I am grateful to Mihai Pascu for the permission to reuse material from his articles in these notes.

I would like to express my gratitude to Donald Marshall for very helpful discussions.

I thank Laurent Serlet, the organizer of the Saint-Flour Probability Summer School 2013, and the Scientific Board for the invitation to deliver the lectures. My research was generously supported by the National Science Foundation via Grant DMS-1206276.

Seattle, WA
2013

Krzysztof Burdzy

Notation

The notation will not be perfectly consistent throughout these notes. The following list contains some of the most frequently used symbols.

The sets of all real and complex numbers will be denoted $\mathbb{R}$ and $\mathbb{C}$. The real and imaginary parts of x will be denoted $\operatorname{Re} x$ and $\operatorname{Im} x$. The Euclidean norm in $\mathbb{R}^d$ will be denoted $|\cdot|$ or $\|\cdot\|$.

An open ball with center x and radius r will be denoted $\mathcal{B}(x, r)$. We will identify points $x \in \mathbb{R}^2$ with vectors $\overrightarrow{(0,0), x}$ and complex numbers $x = re^{i\theta}$. The angle between $x = r_x e^{i\theta_x}$ and $y = r_y e^{i\theta_y}$, i.e., $\theta_x - \theta_y$, will be denoted $\angle(x, y)$. We will use the convention that $\angle(x, y) \in (-\pi, \pi]$.

A function f is called Lipschitz with constant c if $|f(x) - f(y)| \le c|x - y|$ for all x and y. A domain (i.e., an open connected subset of $\mathbb{R}^d$) is called a Lipschitz domain if its boundary can be represented, in a neighborhood of every boundary point, as the graph of a Lipschitz function in some orthonormal coordinate system. The inward unit normal vector at a boundary point $x \in \partial D$ of a domain D will be denoted $\mathbf{n}(x)$ (provided it exists).

For any process Z_t, we will denote the hitting time of a set A by T_A^Z or $T^Z(A)$, i.e., $T_A^Z = \inf\{t \ge 0 : Z_t \in A\}$. The superscript will be dropped if no confusion may arise. The exit time from a set A will be denoted τ_A, that is, $\tau_A = \tau_A^Z = \inf\{t \ge 0 : Z_t \notin A\}$.

Brownian motion will be typically denoted B_t or W_t. Reflected Brownian motion will be often denoted X_t or Y_t.

The distribution of Brownian motion B starting from x will be denoted $\mathbb{P}^x$. For a probability measure μ on $\mathbb{R}$, $\mathbb{P}^\mu$ will denote Brownian motion with the initial distribution equal to μ. The corresponding expectations will be denoted $\mathbb{E}^x$ and $\mathbb{E}^\mu$.

Typically, $\varphi_1, \varphi_2, \dots$ will denote eigenfunctions of the Laplacian in a Euclidean domain (with Neumann or some other boundary conditions). The eigenvalues will be denoted by $\mu_1 \le \mu_2 \le \mu_3 \dots$, with the convention that $\mu_k \ge 0$.

Contents