Lecture Notes in Mathematics 2104

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Twisted Teichmüller Curves

Springer

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ISBN 978-3-319-04074-5 ISBN 978-3-319-04075-2 (eBook)
DOI 10.1007/978-3-319-04075-2
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2014931374

Mathematics Subject Classification (2010): 14G35, 20H10, 11R11, 37D40

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To my parents

Preface

Let X_D denote the Hilbert modular surface $\mathbb{H} \times \mathbb{H}^- / \mathrm{SL}_2(\mathcal{O}_D)$, where $\mathbb{H}$ is the complex upper half plane, $\mathbb{H}^-$ is the complex lower half plane, $\mathcal{O}_D$ is a real quadratic order of discriminant D, and $\mathrm{SL}_2(\mathcal{O}_D)$ acts on $\mathbb{H} \times \mathbb{H}^-$ by the Möbius transformations given by the two embeddings $\mathrm{SL}_2(\mathcal{O}_D) \to \mathrm{SL}_2(\mathbb{R})$. Originally, the interest in Hilbert modular surfaces arose from Hilbert's aim to find an analytic function which plays the same role for arbitrary algebraic number fields as the exponential function does in Kronecker's Theorem about abelian extensions of $\mathbb{Q}$ (compare [vdG88]).

Later, algebraic curves on Hilbert modular surfaces came more into the focus of research. This was triggered by an astonishing result of Hirzebruch and Zagier: they introduced twisted diagonals, which are nowadays also called Hirzebruch–Zagier cycles, via maps $\mathbb{H} \to \mathbb{H} \times \mathbb{H}^-$ with $z \mapsto (Mz, -M^\sigma z)$, where $M \in \mathrm{GL}_2^+(K)$ and σ denotes the Galois conjugate, and they found out that the intersection numbers of certain twisted diagonals can be interpreted as the Fourier coefficients of holomorphic elliptic modular forms of weight two (see [HZ76]). This is the main reason why twisted diagonals have been extensively treated in the literature. For instance, it is well known how to calculate the volume of twisted diagonals.

The projection of a twisted diagonal to X_D yields a Kobayashi curve, i.e., an algebraic curve which is a geodesic for the Kobayashi metric on X_D. So far, there have been found only few examples of Kobayashi curves on X_D that do not stem from twisted diagonals. These rare examples come from Teichmüller curves and have been constructed implicitly by Calta and McMullen. Teichmüller curves are algebraic curves in the moduli space of Riemann surfaces $\mathcal{M}_g$, which are geodesic for the Kobayashi metric. Under the Torelli map the image of such a Teichmüller curve constructed by Calta and McMullen lies in X_D and yields a Kobayashi curve. This implies that in the universal cover the curve is of the form $z \mapsto (z, \varphi(z))$ for some holomorphic map φ.

*The author is partially supported by the ERC-StG 257137.

In these notes, a new class of algebraic curves on Hilbert modular surfaces is introduced: a possibility to construct even more Kobayashi curves on X_D is to consider the projection of $z \mapsto (Mz, M^\sigma \varphi(z))$ to X_D where again $M \in \mathrm{GL}_2^+(K)$. These curves are called twisted Teichmüller curves, because their construction is very reminiscent of Hirzebruch–Zagier cycles. These new objects are analyzed in detail in these notes and their main properties are described. In particular, the volume of twisted Teichmüller curves is calculated and their components are partially classified.

Structure of the Notes

These notes are organized as follows:

Chapter 1 is the introduction and summarizes the results proven in the following chapters.

In Chap. 2, we give an overview of the basic concepts which are used everywhere else in these notes. We recall well-known results about real quadratic number fields, Fuchsian groups, moduli spaces, and Hilbert modular surfaces. This chapter will be helpful in particular for people who are not already familiar with the mentioned topics.

In Chap. 3, we introduce Teichmüller curves. In particular, we recall Calta's and McMullen's construction of Teichmüller curves in $\mathcal{M}_2$, which was already mentioned above.

In Chap. 4, the main new objects of these notes, namely twisted Teichmüller curves, are defined. Only some main properties of twisted Teichmüller curves are derived here. Most importantly, it is shown that twisted Teichmüller curves yield indeed Kobayashi curves.

In Chap. 5, we describe the relation between the stabilizer of the graph of the Teichmüller curve and the commensurator of the Veech group. Furthermore we introduce the notion of pseudo parabolic maximal Fuchsian groups and show why this property is useful for calculating the stabilizer.

In Chap. 6, the volume of twisted Teichmüller curves is calculated for most matrices M if the class number of $\mathcal{O}_D$ is equal to 1. From this the classification of twisted Teichmüller curves can be derived. Finally, we present some ideas how quantities like the number of elliptic fixed points and the number of cusps and the genus of twisted Teichmüller curves can be calculated in some special cases.

In Chap. 7, we recall McMullen's construction of Teichmüller curves in $\mathcal{M}_3$ and $\mathcal{M}_4$ using Prym varieties. Also these Teichmüller curves yield Kobayashi curves on X_D.

In Chap. 8, we recall Oseledet's Theorem on the existence of Lyapunov exponents and introduce the Kontsevich–Zorich cocycle over Teichmüller curves. Furthermore the connection of the Teichmüller flow to the geodesic flow on $T^1\mathbb{H}$ is discussed.

In Chap. 9, it is proven by using Lyapunov exponents that the Teichmüller curves from Chap. 8 are never twists of Teichmüller curves in $\mathcal{M}_2$.

Acknowledgments

These notes are the expanded version of my Ph.D. thesis at the Goethe Universität Frankfurt am Main. First and foremost I would like to thank my advisor Prof. Dr. Martin Möller for his continuous and motivating support during all stages of this work. Furthermore I am very grateful to my second advisor Prof. Dr. Don Zagier for his idea to consider twisted Teichmüller curves at all and for helpful discussions. Many thanks go to Dr. Anke Pohl, who was always willing to answer my mathematical question, to Prof. Dr. Kai-Uwe Bux, who helped me to work out Appendix A.3, and to my colleagues in Bonn in Frankfurt, in particular to my office mates Soumya Bhattacharya, Dr. Tobias Fritz, and Quentin Gendron. I also thank Prof. Dr. Curtis McMullen and JProf. Dr. Gabriela Weitze-Schmithüsen for useful comments on an earlier version of these notes. Moreover, I want to express my gratitude to the referees for their appropriate and constructive suggestions and for proposed corrections to improve these notes. Parts of the work on these notes were carried out at the Max-Planck-Institute for Mathematics in Bonn, whom I thank for hospitality and an inspiring scientific atmosphere. Furthermore, the author was partially supported by the ERC-StG 257137. Last but not least, I would like to thank my friends, my family, and in particular my parents for always having a sympathetic ear for me.

Frankfurt am Main, Deutschland — Christian Weiß
November 2013

List of Symbols

K	Real quadratic number field $\mathbb{Q}(\sqrt{D})$, 11
$\mathrm{N}(\cdot)$	Norm of an element in K, 12
$\mathcal{N}(\cdot)$	Absolute value of the norm of an element in K, 12
$\mathrm{tr}(\cdot)$	Trace of an element in K, 12
$\mathcal{O}_D$	Ring of integers in K, 12 resp. real quadratic order, 14
w	Second basis element of $\mathcal{O}_D$, 12, 14
$\mathcal{O}_D^*$	Units in $\mathcal{O}_D$, 12
ϵ	Fundamental unit, 12
$\mathcal{O}_D^\vee$	Inverse different, 12
h_D	Class number of $\mathcal{O}_D$, 12
(a)	Fractional ideal generated by $a \in K^*$, 12
$\mathfrak{p}$	Prime ideal, 13
$\left(\frac{D}{p}\right)$	Legendre symbol, 14
$\mathbb{H}$	Complex upper half plane, 16
$\mathbb{P}^1(\mathbb{C})$	Projective plane, 16
$\overline{\mathbb{H}}$	Closure of $\mathbb{H}$, 16
$\mu(\cdot)$	Hyperbolic area, 16
$\mathbb{D}$	Unit disc, 16
$\mathrm{SL}_2(\cdot)$	Special linear group, 16
$\mathrm{PSL}_2(\mathbb{R})$	$\mathrm{SL}_2(\mathbb{R})/\pm\mathrm{Id}$, 16
Γ	Fuchsian group, 17
$\mathrm{tr}(\cdot)$	Trace of a matrix, 17
$\mathcal{F}$	Fundamental domain, 17
Γ^M	$M^{-1}\Gamma M$, 17
$(g; m_1, \ldots m_r; s)$	Signature of a Fuchsian group, 17
$\chi(\cdot)$	Euler characteristic, 18

$[A : B]$	Index of a subgroup B in A, 18
$\mathrm{Comm}_G(A)$	Commensurator of A in G, 18
$\Gamma^D(n)$	Principal congruence subgroup, 19
$\Gamma_0^D(n)$	Hecke congruence subgroup, 21
$\Gamma^{D,0}(n)$	Hecke congruence subgroup, 21
$\Gamma^D(m,n)$	$\Gamma_0^D(m) \cap \Gamma^{D,0}(n)$, 21
(X,ω)	Flat surface, 23
$\mathrm{SL}(X,\omega)$	Veech group of a flat surface, 23
$\mathcal{T}(S), \mathcal{T}_g$	Teichmüller space, 25
$\mathrm{Mod}(S)$	Mapping class group, 25
$\mathcal{M}_g$	Moduli space of compact Riemann surfaces of genus g, 25
$\overline{\mathcal{M}_g}$	Deligne-Mumford compactification of $\mathcal{M}_g$, 25
$\Omega(X)$	Nonzero holomorphic $1-$forms on X, 25
$\Omega\mathcal{M}_g$	Set of flat surfaces, 25
$\Omega\mathcal{M}_g(k_1,\dots,k_n)$	Stratum of $\Omega\mathcal{M}_g$, 27
$H^i(X,S)$	i-th Cohomology of X with coefficients in S, 27
$H_i(X,S)$	i-th Homology of X with coefficients in S, 27
$\epsilon(X,\omega)$	Spin invariant, 27
$\Omega\mathcal{M}_g{}^{\mathrm{hyp}}(\cdot)$, $\Omega\mathcal{M}_g{}^{\mathrm{odd}}(\cdot)$, $\Omega\mathcal{M}_g{}^{\mathrm{even}}(\cdot)$	Connected components of $\Omega\mathcal{M}_g(\cdot)$, 28
Λ	Lattice, 29
$c_1(L)$	Chern class of the line bundle L, 29
Π	Period matrix, 30
$\mathrm{Jac}(X)$	Jacobian of X, 30
$\mathrm{End}(X)$	Endomorphisms of X, 30
ρ	Real multiplication, 30
$\mathbb{H}_g$	Siegel upper half space, 30
$\mathcal{A}_g^D$	Moduli space of Abelian varieties of polarization D, 31
$\mathcal{A}_g$	Moduli space of principally polarized Abelian varieties, 31
$\mathrm{SL}(\mathcal{O}_D \oplus \mathfrak{a})$	Hilbert modular group with respect to $\mathfrak{a}$, 32
X_D	Hilbert modular surface, 32
$G(M,V)$	Type of a cusp, 33
ζ_K	Zeta-function of K, 33
$\sigma_1(a)$	Sum of divisors of $a \in \mathbb{Z}$, 34
$\mathrm{GL}_2^+(K)$	Group of matrices with totally positive determinant, 35
M^σ	Galois conjugate of a Matrix in $\mathrm{GL}_2^+(K)$, 35
T_N	Curve on X_D, 36

P_D	Reducible locus in X_D, 36
$k_W(\cdot,\cdot)$	Kobayashi metric, 36
$g(X)$	Genus of Riemann surface X, 41
$O_2(\mathbb{R})$	Orthogonal group, 42
$P(a,b)$	L-shaped polygon, 42
$\mathrm{SL}(L_D^1)$	Veech group of an odd spin L-shaped polygon of discriminant D, 42
$C_{L,D}^{\epsilon}$	Teichmüller curve in $\mathcal{M}_2$ with discriminant D and spin ϵ, 43
$\mathrm{SL}(L_D^0)$	Veech group of an even spin L-shaped polygon of discriminant D, 44
$\mathrm{SL}(L_D)$	Veech group of an arbitrary L-shaped polygon of discriminant D, 44
$\Phi(z) = (z,\varphi(z))$	Graph of the Teichmüller curve, 44
T	$\begin{pmatrix}1 & w\\0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & w-1\\0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & w+1\\0 & 1\end{pmatrix}$, 46, 48, 50
S	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$, 46
Z	$\begin{pmatrix}1 & 0\\w & 1\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\w+1 & 1\end{pmatrix}$, 46, 48, 50
$\mathrm{Stab}(\Phi)$	Stabilizer of the graph of a Teichmüller curve, 54
C_M	Twisted Teichmüller curve, 54
$\mathrm{SL}_M(L_D)$	Stabilizer of the graph of the twisted Teichmüller curve, 54
$\mathrm{SL}_2(\mathcal{O}_D, M)$	$\mathrm{SL}_2(\mathcal{O}_{\mathcal{D}}) \cap M^{-1}\mathrm{SL}_2(\mathcal{O}_D)M$, 55
$X_D(M)$	Level covering of X_D, 55
$\mathrm{SL}(L_D, M)$	$\mathrm{SL}(L_D) \cap M\mathrm{SL}_2(\mathcal{O}_D)M^{-1}$, 55
$C(M)$	$\mathbb{H}/\mathrm{SL}(L_D, M)$, 55
$\mathrm{SL}_M(L_D, M)$	$\mathrm{SL}_M(L_D) \cap M\mathrm{SL}_2(\mathcal{O}_D)M^{-1}$, 55
$C_M(M)$	$\mathbb{H}/\mathrm{SL}_M(L_D, M)$, 55
$\mathrm{SL}^M(L_D)$	$\mathrm{SL}_M(L_D)^M$, 56
$\mathrm{SL}^M(L_D, M)$	$\mathrm{SL}_M(L_D, M)^M$, 56
$C^M(M)$	$\mathbb{H}/\mathrm{SL}^M(L_D, M)$, 56
$\mathfrak{p}_2$	Common prime ideal divisor of (2) and (w), 71
$\mathrm{SL}_2^{tr}(K)$	$\{x \in \mathrm{SL}_2(K) \mid \mathrm{tr}(x) \in \mathcal{O}_D\}$, 79
η^+	Upper right entry of T, 86
η^-	Lower left entry of Z, 86
η^*	Least common multiple of η^+ and η^-, 86
$\mathrm{St}_{\mathrm{SL}_2(\mathbb{R})^2}(C)$	Twisting stabilizer of Teichmüller curve C in $\mathrm{SL}_2(\mathbb{R})^2$, 108
$\mathrm{St}_{\mathrm{GL}_2^+(K)}(C)$	Twisting stabilizer of Teichmüller curve C in $\mathrm{GL}_2^+(K)$, 108
$\mathbb{P}^1(\cdot)$	Projective space, 115

$\Omega(X)^+$	Even Abelian differentials, 122
$\Omega(X)^-$	Odd Abelian differentials, 122
$\mathrm{Prym}(X', \rho)$	Prym variety of (X', ρ), 122
ΩE_D^g	Prym eigenspace, 122
ΩW_D^g	Weierstrass locus, 122
(X, μ)	Probability space, 127
$L^1(X, \mu)$	Space of integrable, measurable functions $f : X \to \mathbb{R}$ with respect to μ, 127
λ_i	i-th Lyapunov exponent, 129
l_i	Multiplicity of the i-th Lyapunov exponent, 129
$T_z\mathbb{H}$	Tangent plane of $\mathbb{H}$ at z, 129
$T^1\mathbb{H}$	Tangent space of $\mathbb{H}$, 129
a_t	Geodesic flow, 130
g_t	Teichmüller flow, 130
$\Omega\mathcal{M}_g{}^{(1)}$	Flat surfaces with renormalized area, 130
$d\nu_1$	Masur-Veech-measure, 130
$\mathcal{Q}(d_1, \ldots, d_n)$	Stratum of meromorphic differentials, 143

Contents