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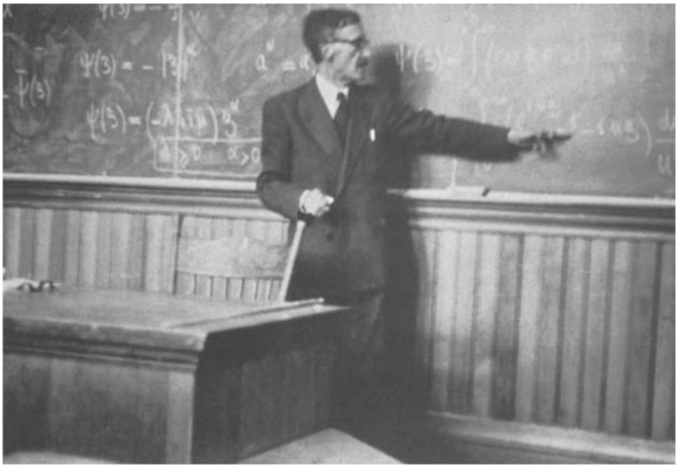

"Lévy Matters" is a subseries of the Springer Lecture Notes in Mathematics, devoted to the dissemination of important developments in the area of Stochastics that are rooted in the theory of Lévy processes. Each volume will contain state-of-the-art theoretical results as well as applications of this rapidly evolving field, with special emphasis on the case of discontinuous paths. Contributions to this series by leading experts will present or survey new and exciting areas of recent theoretical developments, or will focus on some of the more promising applications in related fields. In this way each volume will constitute a reference text that will serve PhD students, postdoctoral researchers and seasoned researchers alike.

The volumes in this subseries are published under the auspices of the Bernoulli Society.

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Lévy Matters III

Lévy-Type Processes: Construction, Approximation and Sample Path Properties

Springer

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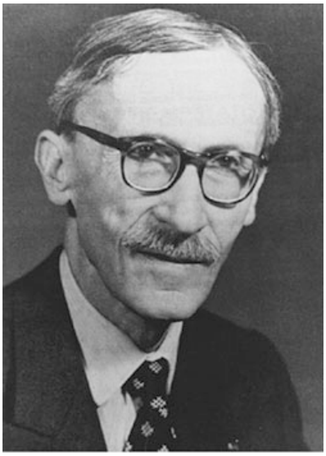
Preface to the Series Lévy Matters



Over the past 10–15 years, we have seen a revival of general Lévy processes theory as well as a burst of new applications. In the past, Brownian motion or the Poisson process had been considered as appropriate models for most applications. Nowadays, the need for more realistic modelling of irregular behaviour of phenomena in nature and society like jumps, bursts and extremes has led to a renaissance of the theory of general Lévy processes. Theoretical and applied researchers in fields as diverse as quantum theory, statistical physics, meteorology, seismology, statistics, insurance, finance and telecommunication have realized the enormous flexibility of Lévy models in modelling jumps, tails, dependence and sample path behaviour. Lévy processes or Lévy-driven processes feature slow or rapid structural breaks, extremal behaviour, clustering and clumping of points.

Tools and techniques from related but distinct mathematical fields, such as point processes, stochastic integration, probability theory in abstract spaces and differential geometry, have contributed to a better understanding of Lévy jump processes.

As in many other fields, the enormous power of modern computers has also changed the view of Lévy processes. Simulation methods for paths of Lévy processes and realizations of their functionals have been developed. Monte Carlo simulation makes it possible to determine the distribution of functionals of sample paths of Lévy processes to a high level of accuracy.

This development of Lévy processes was accompanied and triggered by a series of Conferences on Lévy Processes: Theory and Applications. The First and Second Conferences were held in Aarhus (1999, 2002), the Third in Paris (2003), the Fourth in Manchester (2005) and the Fifth in Copenhagen (2007).

To show the broad spectrum of these conferences, the following topics are taken from the announcement of the Copenhagen conference:

- Structural results for Lévy processes: distribution and path properties
- Lévy trees, superprocesses and branching theory

- Fractal processes and fractal phenomena
- Stable and infinitely divisible processes and distributions
- Applications in finance, physics, biosciences and telecommunications
- Lévy processes on abstract structures
- Statistical, numerical and simulation aspects of Lévy processes
- Lévy and stable random fields

At the Conference on Lévy Processes: Theory and Applications in Copenhagen the idea was born to start a series of Lecture Notes on Lévy processes to bear witness of the exciting recent advances in the area of Lévy processes and their applications. Its goal is the dissemination of important developments in theory and applications. Each volume will describe state-of-the-art results of this rapidly evolving subject with special emphasis on the non-Brownian world. Leading experts will present new exciting fields, or surveys of recent developments, or focus on some of the most promising applications. Despite its special character, each article is written in an expository style, normally with an extensive bibliography at the end. In this way each article makes an invaluable comprehensive reference text. The intended audience are Ph.D. and postdoctoral students, or researchers, who want to learn about recent advances in the theory of Lévy processes and to get an overview of new applications in different fields.

Now, with the field in full flourish and with future interest definitely increasing it seemed reasonable to start a series of Lecture Notes in this area, whose individual volumes will appear over time under the common name "Lévy Matters," in tune with the developments in the field. "Lévy Matters" appears as a subseries of the Springer Lecture Notes in Mathematics, thus ensuring wide dissemination of the scientific material. The mainly expository articles should reflect the broadness of the area of Lévy processes.

We take the possibility to acknowledge the very positive collaboration with the relevant Springer staff and the editors of the LN series and the (anonymous) referees of the articles.

We hope that the readers of "Lévy Matters" enjoy learning about the high potential of Lévy processes in theory and applications. Researchers with ideas for contributions to further volumes in the Lévy Matters series are invited to contact any of the editors with proposals or suggestions.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
Paris, France — Jean Bertoin and Jean Jacod
Munich, Germany — Claudia Klüppelberg
June 2010

A Short Biography of Paul Lévy

A volume of the series "Lévy Matters" would not be complete without a short sketch about the life and mathematical achievements of the mathematician whose name has been borrowed and used here. This is more a form of tribute to Paul Lévy, who not only invented what we call now Lévy processes, but also is in a sense the founder of the way we are now looking at stochastic processes, with emphasis on the path properties.

Paul Lévy was born in 1886 and lived until 1971. He studied at the Ecole Polytechnique in Paris and was soon appointed as professor of mathematics in the same institution, a position that he held from 1920 to 1959. He started his career as an analyst, with 20 published papers between 1905 (he was then 19 years old) and 1914, and he became interested in probability by chance, so to speak, when asked to give a series of lectures on this topic in 1919 in that same school: this was the starting point of an astounding series of contributions in this field, in parallel with a continuing activity in functional analysis.

Very briefly, one can mention that he is the mathematician who introduced characteristic functions in full generality, proving in particular the characterization theorem and the first "Lévy's theorem" about convergence. This naturally led him to study more deeply the convergence in law with its metric and also to consider sums of independent variables, a hot topic at the time: Paul Lévy proved a form of the 0-1 law, as well as many other results, for series of independent variables. He also introduced stable and quasi-stable distributions and unravelled their weak and/or strong domains of attractions, simultaneously with Feller.

Then we arrive at the book *Théorie de l'addition des variables aléatoires*, published in 1937, and in which he summaries his findings about what he called "additive processes" (the homogeneous additive processes are now called Lévy processes, but he did not restrict his attention to the homogeneous case). This book contains a host of new ideas and new concepts: the decomposition into the sum of jumps at fixed times and the rest of the process; the Poissonian structure of the jumps for an additive process without fixed times of discontinuities; the "compensation" of those jumps so that one is able to sum up all of them; the fact that the remaining continuous part is Gaussian. As a consequence, he implicitly gave the formula

providing the form of all additive processes without fixed discontinuities, now called the Lévy–Itô formula, and he proved the Lévy–Khintchine formula for the characteristic functions of all infinitely divisible distributions. But, as fundamental as all those results are, this book contains more: new methods, like martingales which, although not given a name, are used in a fundamental way; and also a new way of looking at processes, which is the "pathwise" way: he was certainly the first to understand the importance of looking at and describing the paths of a stochastic process, instead of considering that everything is encapsulated into the distribution of the processes.

This is of course not the end of the story. Paul Lévy undertook a very deep analysis of Brownian motion, culminating in his book *Processus stochastiques et mouvement Brownien* in 1948, completed by a second edition in 1965. This is a remarkable achievement, in the spirit of path properties, and again it contains so many deep results: the Lévy modulus of continuity, the Hausdorff dimension of the path, the multiple points and the Lévy characterization theorem. He introduced local time and proved the arc-sine law. He was also the first to consider genuine stochastic integrals, with the area formula. In this topic again, his ideas have been the origin of a huge amount of subsequent work, which is still going on. It also laid some of the basis for the fine study of Markov processes, like the local time again, or the new concept of instantaneous state. He also initiated the topic of multi-parameter stochastic processes, introducing in particular the multi-parameter Brownian motion.

As should be quite clear, the account given here does not describe the whole of Paul Lévy's mathematical achievements, and one can consult for many more details the first paper (by Michel Loève) published in the first issue of the *Annals of Probability* (1973). It also does not account for the humanity and gentleness of the person Paul Lévy. But I would like to end this short exposition of Paul Lévy's work by hoping that this series will contribute to fulfilling the program, which he initiated.

Paris, France Jean Jacod

Preface

Behind every decent Markov process there is a family of Lévy processes. Indeed, let $(X_t)_{t\geqslant 0}$ be a Markov process with state space $\mathbb{R}^d$ and assume, for the moment, that the limit

$$\lim_{t\to 0} \frac{1 - \mathbb{E}^x\, e^{i\xi\cdot(X_t - x)}}{t} = q(x,\xi) \quad \forall x,\xi \in \mathbb{R}^d \tag{$\bigstar$}$$

exists such that the function $\xi \mapsto q(x,\xi)$ is continuous. We will see below that this is enough to guarantee that $q(x,\cdot)$ is, for each $x \in \mathbb{R}^d$, the characteristic exponent of a Lévy process; as such, it enjoys a Lévy–Khintchine representation

$$q(x,\xi) = -il(x)\cdot\xi + \frac{1}{2}\xi\cdot Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} \big(1 - e^{iy\cdot\xi} + i\xi\cdot y\mathbb{1}_{[-1,1]}(|y|)\big)\, N(x,dy)$$

where $(l(x), Q(x), N(x,dy))$ is for every fixed $x \in \mathbb{R}^d$ a Lévy triplet. The function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called the **symbol of the process**. The processes which admit a symbol behave locally like a Lévy process, and their infinitesimal generators resemble the generators of Lévy processes *with variable, i.e. x-dependent, coefficients*—this justifies the name **Lévy-type processes**. This guides us to the main topics of the present tract:

Characterization:	For which Markov processes does the limit ($\bigstar$) exist?
Construction:	Is there a Lévy-type process with a given symbol $q(x,\xi)$? Is there a 1-to-1 correspondence between symbols and processes?
Sample paths:	Can we use the symbol $q(x,\xi)$ in order to describe the sample path behaviour of the process?
Approximation:	Is it possible to use $q(x,\xi)$ to approximate and to simulate the process?

Let us put this point of view into perspective by considering first of all some d-dimensional Lévy process $(X_t)_{t\geqslant 0}$. Being a (strong) Markov process, $(X_t)_{t\geqslant 0}$ can

be described by the transition function $p_t(x, dy) = \mathbb{P}^x(X_t \in dy) = \mathbb{P}(X_t + x \in dy)$ which, in turn, is uniquely characterized by the characteristic function

$$\mathbb{E}^x\, e^{i\xi\cdot X_t} = \int_{\mathbb{R}^d} e^{i\xi\cdot y}\, p_t(x, dy) = e^{i\xi\cdot x} e^{-t\psi(\xi)} \tag{1}$$

and the characteristic exponent ψ. Thus,

$$1 - \int_{\mathbb{R}^d} e^{i\xi\cdot(y-x)}\, p_t(x, dy) = t\psi(\xi) + \mathrm{o}(t) \quad \text{as } t \to 0 \tag{2}$$

and, with some effort, we can derive from this the Lévy–Khintchine representation of the exponent ψ

$$\psi(\xi) = -il\cdot\xi + \frac{1}{2}\xi\cdot Q\xi + \int_{\mathbb{R}^d\setminus\{0\}} \big(1 - e^{iy\cdot\xi} + i\xi\cdot y\mathbb{1}_{[-1,1]}(|y|)\big)\,\nu(dy) \tag{3}$$

where (l, Q, ν) is the Lévy triplet. The key observation is that the family of measures $t^{-1}p_t(x, B + x) = t^{-1}\,\mathbb{P}(X_t \in B)$ converges[1] to the Lévy measure $\nu(B)$ as $t \to 0$ for all Borel sets $B \subset \mathbb{R}^d \setminus \{0\}$ satisfying $\nu(\bar{B} \setminus B^\circ) = 0$ and $0 \notin \bar{B}$.

In our calculation there is only one place where we used Lévy processes: The second equality sign in (1) which is the consequence of the translation invariance (spatial homogeneity) and infinite divisibility of a Lévy process. If we do away with it, and if we only assume that $(X_t)_{t\geqslant 0}$ is strong Markov with transition function $(p_t(x, dy))_{t\geqslant 0,\, x\in\mathbb{R}^d}$, we still have that

$$\lambda_t(x,\xi) := \mathbb{E}^x\, e^{i\xi\cdot(X_t - x)} = \int_{\mathbb{R}^d} e^{i\xi\cdot(y-x)}\, p_t(x, dy). \tag{1'}$$

Assume we *knew* that $t^{-1}p_t(x, B + x)$ has, as $t \to 0$, for every $x \in \mathbb{R}^d$ and suitable Borel sets $B \subset \mathbb{R}^d \setminus \{0\}$, a limit $N(x, B)$ which is a kernel on $\mathbb{R}^d \times \mathscr{B}(\mathbb{R}^d \setminus \{0\})$. Then we would get, as in (2),

$$1 - \lambda_t(x,\xi) = tq(x,\xi) + \mathrm{o}(t) \quad \text{as } t \to 0. \tag{2'}$$

But, what can be said about $q(x,\xi)$?

With some elementary harmonic analysis this can be worked out. Since $\xi \mapsto \lambda_t(x,\xi)$ is a characteristic function, it is continuous and positive definite (see p. 41 for the definition); and since $1 \geqslant \lambda_t(x, 0) \geqslant |\lambda_t(x,\xi)|$, we get from this that

$$\sum_{j,k=1}^{n} \big(\lambda_t(x, \xi_j - \xi_k) - 1\big)\mu_j\bar{\mu}_k \geqslant 0 \tag{4}$$

[1]The classical proofs use here Lévy's continuity theorem or the Helly–Bray theorem.

for all $n \geqslant 0$, $\xi_1, \dots, \xi_n \in \mathbb{R}^d$ and $\mu_1, \dots, \mu_n \in \mathbb{C}$ with $\sum_{j=1}^n \mu_j = 0$. This means that $\xi \mapsto \lambda_t(x, \xi) - 1$ is continuous and *conditionally positive definite*[2] (because of the *condition* $\sum_j \mu_j = 0$, cf. p. 42). The important point is now that every continuous and conditionally positive definite function enjoys a Lévy–Khintchine representation.

Obviously, inequality (4) remains valid if we divide by t and let $t \to 0$, so

$$\lim_{t \to 0} \frac{1 - \mathbb{E}^x e^{i \xi \cdot (X_t - x)}}{t} = q(x, \xi) \quad \forall x, \xi \in \mathbb{R}^d \tag{$\bigstar$}$$

defines a conditionally positive definite function $\xi \mapsto -q(x, \xi)$. If it is also continuous, then it has for every fixed $x \in \mathbb{R}^d$ a Lévy–Khintchine representation, and each $q(x, \cdot)$ is the characteristic exponent of a Lévy process. We will call the function $q(x, \xi)$ the **symbol** of the process $(X_t)_{t \geqslant 0}$. In this sense it is correct to say that *behind every decent Markov process* $(X_t)_{t \geqslant 0}$ *there is a family of Lévy processes* $(L_t^{(x)})_{t \geqslant 0, x \in \mathbb{R}^d}$ whose characteristic exponents are given by ($\bigstar$), and we are back at the point where we started our discussion.

Sufficient conditions for the limit ($\bigstar$) to exist are best described by a list of Lévy-type processes: Lévy processes, of course, (cf. Sect. 2.1), any Feller process whose infinitesimal generator has a sufficiently rich domain (Sects. 2.3 and 2.4), many Lévy-driven stochastic differential equations (Sect. 3.2), or temporally homogeneous Markovian jump-diffusion semimartingales (Sect. 2.5) provided that their extended generator contains sufficiently many functions. As it turns out, the symbol $q(x, \xi)$ encodes, via its Lévy–Khintchine representation and the (necessarily) x-dependent Lévy triplet, the semimartingale characteristics of the stochastic process; moreover, it yields a simple representation of the infinitesimal generator as a pseudo-differential operator.

We are not aware of necessary conditions such that ($\bigstar$) defines a negative definite symbol, although the class of temporally homogeneous Markovian jump-diffusion semimartingales looks pretty much to be the largest class of decent strong Markov processes admitting a symbol.

Most of our results hold for any *decent* strong Markov process admitting a symbol $q(x, \xi)$, but we restrict our attention to Feller processes where *decency* comes from the natural assumption that the compactly supported smooth functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of the infinitesimal generator. The key results in this direction are our short proof of the Courrège–von Waldenfels theorem (Theorem 2.21) and the probabilistic formula for the symbol, Theorems 2.36 and 2.44.

Let us briefly explain how the material is organized. The *Primer on Feller Semigroups*, Chap. 1, is included in order to make the material accessible for the novice and also to serve as a reference. For the more experienced reader, the ideal point of departure should be Sect. 2.1 on Lévy processes which leads directly

[2] Also known as *negative definite*, and we will prefer this notion in the sequel, cf. Sect. 2.2

to the characterization of Feller processes. Among the central results of Chap. 2 is the characterization of the generators as pseudo-differential operators and the fact that Feller processes are semimartingales: In both cases the symbol $q(x,\xi)$ and its x-dependent Lévy triplet are instrumental. Chapter 3 is devoted to various construction methods for Feller processes. This is probably the most technical part of our treatise since techniques from different areas of mathematics come to bearing; it is already difficult to describe the results, to present complete proofs in this essay is near impossible. Nevertheless we tried to describe the ideas how things fit together, and we hope that the interested reader follows up on the references provided. Perturbations and time-changes for Feller processes are briefly discussed in Chap. 4. In particular, we obtain conditions such that the Feller property is preserved under these transformations. From Chap. 5 onwards, things become more probabilistic: Now we show how to use the symbol $q(x,\xi)$ in order to describe the behaviour of the sample paths of a Feller process. For Lévy processes this approach has a long tradition starting with the papers by Blumenthal–Getoor [31,32] in the early sixties; a survey is given in Fristedt [112]. The principal tool for these investigations is probability estimates for the running maximum of the process in terms of the symbol (Sect. 5.1). Using these estimates we can define Blumenthal–Getoor–Pruitt indices for Feller processes which, in turn, allow us to find bounds for the Hausdorff dimension of the sample paths, describe the (polynomial) short- and long-time asymptotics of the paths, their p-variation, their Besov regularity, etc. Returning to the level of transition semigroups we then investigate global properties (in the sense of Fukushima et al.) in Chap. 6. We focus on functional inequalities and their stability under subordination and on coupling methods; the latter are explained in detail for Lévy- and linear Ornstein–Uhlenbeck processes. The classical topics of transience and recurrence are discussed from the perspective of Meyn and Tweedie, with an emphasis on stable-like processes. In the final Chap. 7 we show how the viewpoint of a Feller process being locally Lévy can be used to approximate the sample paths of Feller processes. This allows us, for the first time, to simulate Feller processes with unbounded coefficients. We close this treatise with a list of open problems which we think are important for the further development of the subject.

We cannot cover all aspects of Lévy-type processes in this survey. Notable omissions are probabilistic potential theory, the general theory of Dirichlet forms, heat kernel estimates and processes on domains. Our choice of material was, of course, influenced by personal liking, by our own research interests and by the desire to have a clear focus. Some topics, e.g. probabilistic potential theory and Dirichlet forms, are more naturally set in the wider framework of general Markov processes and there are, indeed, monographs which we think are hard to match: In potential theory there are Chung's books [67,68] (for Feller processes), Sharpe [298] (for general Markov processes), Port–Stone [238] and Bertoin [27] (for Brownian motion and Lévy processes) and for Dirichlet forms there is Fukushima et al. [115] (for the symmetric case), and Ma–Röckner [210] (for the non-symmetric case). Heat kernel estimates are usually discussed in an L^2-framework, cf. Chen [60] for an excellent survey, or for various perturbations of stable Lévy processes (also on domains), e.g. as in Chen–Kim–Song [65, 66]. An interesting geometric approach

has recently been proposed by [168]. Finally, processes on domains with general Wentzell boundary conditions [352] for the generator are still a problem: While some progress has been made in the one-dimensional case (cf. Mandl [216], Langer and co-workers [199, 200]), the multidimensional case is wide open, and the best treatment is Taira [313].

A few words on the style of this treatise are in order. Some time ago, we have been invited to contribute a survey paper to the *Lévy Matters* subseries of the Springer Lecture Notes in Mathematics, updating the earlier paper *Lévy-type processes and pseudo-differential operators* by N. Jacob and one of the present authors. Soon, however, it became clear that the developments in the past decade have been quite substantial while, on the other hand, much of the material is scattered throughout the literature and that a comprehensive treatise on Feller processes is missing. With this essay we try to fill this gap, providing a reliable source for reference (especially for those elusive *folklore* results), making a technically demanding area easily accessible to future generations of researchers and, at the same time, giving a snapshot of the state-of-the-art of the subject. Just as one would expect in a survey, we do not always (want to) give detailed proofs, but we provide precise references whenever we omit proofs or give only a rough outline of the argument (sometimes also sailing under the nickname "proof"). On the other hand, quite a few theorems are new or contain substantial improvements of known results, and in all those cases we do include full proofs or describe the necessary changes to the literature. We hope that the exposition is useful for and accessible to anyone with a working knowledge of Lévy- or continuous-time Markov processes and some basic functional analysis.

It is a pleasure to acknowledge the support of quite a few people. Niels Jacob has our best thanks, his ideas run through the whole text, and we shall think it a success if it pleases him.

Without the named (and, as we fear, often unnamed) contributions of our co-authors and fellow scientists such a survey would not have been possible; we are grateful that we can present and build on their results. Anite Behme, Xiaoping Chen, Katharina Fischer, Julian Hollender, Victorya Knopova, Franziska Kühn, Huaiqian Li, Felix Lindner, Michael Schwarzenberger and Nenghui Zhu read substantial portions of various β-versions of this survey, pointed out many mistakes and inconsistencies, and helped us to improve the text; the examples involving affine processes were drafted by Michael Schwarzenberger.

Special thanks go to Claudia Klüppelberg and the editors of the *Lévy Matters* series for the invitation to write and their constant encouragement to finish this piece.

Finally, we thank our friends and families who—we are pretty sure of it—are more than happy that this work has come to an end.

Dresden, Germany — Björn Böttcher and René Schilling
Fuzhou, China — Jian Wang
April 2013

Contents

Summary of Notation

This list is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below; numbers following an entry are page numbers.

Unless otherwise stated, functions are real-valued, and binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limiting relations $f_j \xrightarrow{j\to\infty} f$, $\lim_j f_j$, $\underline{\lim}_j f_j$, $\overline{\lim}_j f_j$, $\sup_j f_j$ or $\inf_j f_j$ are understood pointwise. "Positive" and "negative" always means "$\geqslant 0$" and "$\leqslant 0$".

General Notation: Analysis

$a \vee b$, $a \wedge b$	$\max(a,b)$, $\min(a,b)$
a^+, a^-	$\max(a,0)$, $-\min(a,0)$
$f \asymp g$	$\exists c\ \forall t\ :\ \frac{1}{c}g(t) \leqslant f(t) \leqslant cg(t)$
$\lfloor x \rfloor$	Largest integer $n \leqslant x$
$\lvert x \rvert$	Euclidean vector and matrix norm
$x \cdot y$, $\langle x, y\rangle$	Scalar product in $\mathbb{R}^d$, $\sum_{j=1}^d x_j y_j$
$\mathbb{1}_A$	$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$
$e_\xi(x)$	$e^{i\xi\cdot x}$, $x, \xi \in \mathbb{R}^d$
δ_x	Point mass at x
$\operatorname{supp} f$	Support, $\overline{\{f \neq 0\}}$
∇	Gradient $\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_d}\right)^\top$
∇^α	$\frac{\partial^{\alpha_1+\dots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$
$\mathcal{F}u$, $\hat{u}$	Fourier transform, 31
$\mathcal{F}^{-1}u$, $\check{u}$	Inverse Fourier transform, 31
bp-lim	Bounded pointwise (bp) convergence, 1
$(A, \mathcal{D}(A))$	Generator, 18, 23
$\hat{A}$, $\hat{A}_b$	Full generator, 25

$\psi(D)$	Fourier multiplier, 37, 51
$q(x, D)$	Pseudo-differential operator, 51

General Notation: Probability

$\sim$	"Is distributed as"
a.s.	Almost surely
$(X_t, \mathscr{F}_t)_{t\geqslant 0}$	Adapted process
(l, Q, ν)	Lévy triplet, 33
χ	Truncation function, 33
τ_r^x	$\inf\{t > 0 : X_t \in \overline{\mathbb{B}}^c(x, r)\}$

Sets and σ-Algebras

A^c	Complement of the set A
A°	Open interior of the set A
$\overline{A}$	Closure of the set A
$\mathbb{B}(x, r)$	Open ball, centre x, radius r
$\overline{\mathbb{B}}(x, r)$	Closed ball, centre x, radius r
$\mathscr{B}(E)$	Borel sets of E
$\mathscr{F}_t^X$	$\sigma(X_s : s \leqslant t)$

Spaces of Measures and Functions

$\\|u\\|_{(k)}$	$\sum_{0\leqslant\|\alpha\|\leqslant k} \\|\nabla^\alpha u\\|_\infty$
$\\|\mu\\|_{TV}$	Total variation norm
$B(E)$	Borel functions on E
$B_b(E)$	——, bounded
$C(E)$	Continuous functions on E
$C_b(E)$	——, bounded
$C_\infty(E)$	——, $\lim_{\|x\|\to\infty} u(x) = 0$
$C_c(E)$	——, compact support
$C^k(E)$	k times continuously diff'ble functions on E
$C_b^k(E)$	——, bounded (with their derivatives)
$C_\infty^k(E)$	——, 0 at infinity (with their derivatives)
$C_c^k(E)$	——, compact support
$L^p(E, \mu), L^p(\mu), L^p(E)$	L^p space w.r.t. the measure space $(E, \mathcal{F}, \mu)$
$\mathcal{M}(E)$	(Signed) Radon measures on E
$\mathcal{M}_b(E)$	——, with finite mass
$\mathcal{M}^+(E)$	——, positive
$\mathcal{M}^1(E)$	Probability measures on E
$\mathcal{S}(\mathbb{R}^d)$	Schwartz space of rapidly decreasing smooth functions