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Cohomological Aspects in Complex Non-Kähler Geometry

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In ricordo di mio nonno,
il Maestro Giovanni

Introduction

By a remarkable result by W. L. Chow, [Cho49, Theorem V], see also [Ser56], *projective manifolds* (that is, compact complex submanifolds of $\mathbb{CP}^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/(\mathbb{C} \setminus \{0\})$, for $n \in \mathbb{N}$) are in fact *algebraic* (that is, they can be described as the zero set of finitely many homogeneous holomorphic polynomials). One is hence interested in relaxing the projective condition, looking for special properties on compact manifolds sharing a weaker structure than projective manifolds. For example, a large amount of developed analytic techniques allows to prove strong cohomological properties for compact *Kähler manifolds* (that is, compact complex manifolds endowed with a Kähler metric, namely, a Hermitian metric admitting a local potential function), [SvD30, Käh33], see also [Wei58], which are, in a certain sense, the "analytic-versus-algebraic", [Cho49, Theorem V], or the "$\mathbb{R}$-versus-$\mathbb{Q}$", [Kod54, Theorem 4], version of projective manifolds. On the one side, there is the class of Kähler manifolds, which are in fact endowed with three different structures, interacting each other: a *complex* structure, a *symplectic* structure, and a *metric* structure; it is the strong linking between them that allows to develop many analytic tools and hence to derive the very special properties of Kähler manifolds. On the other side (the Dark Side…), in order to further investigate any of such properties and to understand what of these three structures is actually involved and required, it is natural to look for manifolds not admitting any Kähler structure: a large amount of interesting non-Kähler manifolds has been provided since [Thu76]. In other words, one is led to study complex, symplectic, and metric contributions separately, possibly weakening either the interactions between them, or one of these structures. For example, by relaxing the metric condition, one could ask what properties of a compact complex manifold can be deduced by the existence of special Hermitian metrics defined by conditions similar to, but weaker than, the defining condition of the Kähler metrics (for example, metrics being *balanced* in the sense of M. L. Michelsohn [Mic82], *pluriclosed* [Bis89], *astheno-Kähler* [JY93, JY94], *Gauduchon* [Gau77], *strongly-Gauduchon* [Pop13]); by relaxing the complex structure, one is led to study properties of *almost-complex* manifold, possibly endowed with compatible symplectic structures.

In these notes, we are concerned with summarizing some recent results on the cohomological properties of compact complex manifolds endowed with no Kähler structure. Cohomological aspects of manifolds endowed with almost-complex structures, or with other special structures (such as, for example, symplectic, generalized-complex, ...) are also considered.

We recall that a *complex* manifold X is endowed with a natural *almost-complex* structure, that is, an endomorphism $J \in \mathrm{End}(TX)$ of the tangent bundle of X such that $J^2 = -\mathrm{id}_{TX}$, which actually satisfies a further integrability condition, [NN57, Theorem 1.1]. By considering the decomposition into eigen-spaces, just the datum of the almost-complex structure yields a splitting of the complexified tangent bundle, namely,

$$TX \otimes \mathbb{C} \;=\; T^{1,0}X \oplus T^{0,1}X \;,$$

and hence it induces also a splitting of the bundle of complex differential forms, namely,

$$\wedge^{\bullet} X \otimes_{\mathbb{R}} \mathbb{C} \;=\; \bigoplus_{p+q=\bullet} \wedge^{p,q} X \;.$$

Furthermore, on a complex manifold, the integrability condition of such an almost-complex structure yields a further structure on $\wedge^{\bullet,\bullet} X$, namely, a structure of double complex $\left(\wedge^{\bullet,\bullet} X,\, \partial,\, \overline{\partial}\right)$, where ∂ and $\overline{\partial}$ are the components of the $\mathbb{C}$-linear extension of the exterior differential d.

Hence, on a complex manifold X, one can consider both the de Rham cohomology

$$H^{\bullet}_{dR}(X;\mathbb{C}) \;:=\; \frac{\ker \mathrm{d}}{\mathrm{im}\, \mathrm{d}}$$

and the Dolbeault cohomology

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \;:=\; \frac{\ker \overline{\partial}}{\mathrm{im}\, \overline{\partial}} \;;$$

whenever X is compact, the Hodge theory assures that they have finite dimension as $\mathbb{C}$-vector spaces. On a compact complex manifold, in general, no natural map between $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and $H^{\bullet}_{dR}(X;\mathbb{C})$ exists; on the other hand, the structure of double complex of $\left(\wedge^{\bullet,\bullet} X,\, \partial,\, \overline{\partial}\right)$ gives rise to a spectral sequence

$$E_1^{\bullet,\bullet} \;\simeq\; H^{\bullet,\bullet}_{\overline{\partial}}(X) \;\Rightarrow\; H^{\bullet}_{dR}(X;\mathbb{C}) \;,$$

from which one gets the *Frölicher inequality*, [Frö55, Theorem 2]: for every $k \in \mathbb{N}$,

$$\dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C}) \;\leq\; \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;.$$

On a complex manifold, a "bridge" between the Dolbeault and the de Rham cohomology is provided, in a sense, by the *Bott-Chern cohomology*,

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial \cap \ker\overline{\partial}}{\mathrm{im}\,\partial\overline{\partial}} \;,$$

and the *Aeppli cohomology*,

$$H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}} \;.$$

In fact, the identity induces the maps of (bi-)graded $\mathbb{C}$-vector spaces

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\partial}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(X) & \end{tikzcd}$$

which, in general, are neither injective nor surjective.

These cohomology groups have been introduced by R. Bott and S. S. Chern in [BC65], and by A. Aeppli in [Aep65], and studied by several authors in different contexts: among others, B. Bigolin [Big69, Big70], A. Andreotti and F. Norguet [AN71], J. Varouchas [Var86], M. Abate [Aba88], L. Alessandrini and G. Bassanelli [AB96], S. Ofman [Ofm85a,Ofm85b,Ofm88], S. Boucksom [Bou04], J.-P. Demailly [Dem12], M. Schweitzer [Sch07], L. Lussardi [Lus10], R. Kooistra [Koo11], J.-M. Bismut [Bis11b, Bis11a], L.-S. Tseng, and S.-T. Yau [TY11].

We recall that whenever X is compact, the Hodge theory can be performed also for Bott-Chern and Aeppli cohomologies, [Sch07, §2], yielding their finite-dimensionality; more precisely, one has that, on a compact complex manifold X of complex dimension n endowed with a Hermitian metric,

$$H^{\bullet,\bullet}_{BC}(X) \;\simeq\; \tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;\simeq\; \tilde{\Delta}_{A} \;,$$

where $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_{A}$ are 4th order self-adjoint elliptic differential operators; furthermore, the Hodge-$*$-operator associated to any Hermitian metric on X induces an isomorphism $H^{p,q}_{BC}(X) \simeq H^{n-q,n-p}_{A}(X)$, for every $p, q \in \mathbb{N}$.

By the definitions, the map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective if and only if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact: a compact complex manifold fulfilling this property is said to satisfy the *$\partial\overline{\partial}$-Lemma*; see [DGMS75] by P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan, where consequences of the validity of the $\partial\overline{\partial}$-Lemma on the real homotopy type of a compact complex manifold are investigated. When the $\partial\overline{\partial}$-Lemma holds, it turns out that actually all the above maps are isomorphisms, [DGMS75, Lemma 5.15, Remark 5.16, 5.21]: in particular, one gets a decomposition

$$H^{\bullet}_{dR}(X;\mathbb{C}) \simeq \bigoplus H^{\bullet,\bullet}_{\overline{\partial}}(X) \qquad \text{such that} \qquad H^{\bullet_1,\bullet_2}_{\overline{\partial}}(X) \simeq \overline{H^{\bullet_2,\bullet_1}_{\overline{\partial}}(X)} \; .$$

A very remarkable property of compact Kähler manifolds is that they satisfy the $\partial\overline{\partial}$-Lemma, [DGMS75, Lemma 5.11]: this follows from the Kähler identities, which can be proven as a consequence of the fact that the Kähler metrics osculate to order 2 the standard Hermitian metric of $\mathbb{C}^n$ at every point. Therefore, the above decomposition holds true, in particular, for compact Kähler manifolds, [Wei58, Théorème IV.3].

In particular, if X is a compact complex manifold satisfying the $\partial\overline{\partial}$-Lemma, then, for every $k \in \mathbb{N}$,

$$\dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) = \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{BC}(X) \; .$$

In the first chapter, we study cohomological properties of compact complex manifolds, studying in particular the Bott-Chern and Aeppli cohomologies, and their relation with the $\partial\overline{\partial}$-Lemma.

In fact, we prove an *inequality à la Frölicher* for the Bott-Chern and Aeppli cohomologies, which provides also a characterization of the compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma just in terms of the dimensions of the Bott-Chern cohomology groups, [AT13b, Theorem A, Theorem B]; a key tool in the proof of the Frölicher-type inequality relies on exact sequences by J. Varouchas, [Var86]. More precisely, we state the following result.

Theorem (see Theorems 2.13 and 2.14). *Let X be a compact complex manifold. Then, for every $k \in \mathbb{N}$, the following inequality holds:*

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \geq 2 \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \; .$$

Furthermore, the equality

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) = 2 \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})$$

holds for every $k \in \mathbb{N}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma.

Note that the equality $\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) = \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C})$ for every $k \in \mathbb{N}$ (which is equivalent to the degeneration of the Hodge and Frölicher spectral

sequence at the first step, $E_1 \simeq E_\infty$) is not sufficient to let X satisfy the $\partial\overline{\partial}$-Lemma: in some sense, the above result states that the Bott-Chern cohomology, together with its dual, the Aeppli cohomology, encodes "more information" on the double complex $\left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ than just the Dolbeault cohomology.

As a straightforward consequence of the previous theorem, we obtain another proof, see [AT13b, Corollary 2.7], of the following stability result, see [Voi02, Proposition 9.21], [Wu06, Theorem 5.12], [Tom08, §B].

Corollary (see Corollary 2.2). *Satisfying the $\partial\overline{\partial}$-Lemma is a stable property under small deformations of the complex structure, that is, if $\{X_t\}_{t\in B}$ is a complex-analytic family of compact complex manifolds and X_{t_0} satisfies the $\partial\overline{\partial}$-Lemma for some $t_0 \in B$, then X_t satisfies the $\partial\overline{\partial}$-Lemma for every t in an open neighbourhood of t_0 in B.*

The previous results can be generalized to a more algebraic context, see [AT13a]. In particular, one gets applications concerning the cohomologies of symplectic manifolds and, more in general, of generalized complex manifolds, and to the study of the Hard Lefschetz Condition and of the $\mathrm{d}\,\mathrm{d}^{\mathcal{J}}$-Lemma, which we summarize in Appendix: Cohomological Properties of Generalized Complex Manifolds. (See Sects. 1.2 and 1.3 for preliminary results on symplectic and generalized complex structures and on their cohomologies.)

We recall that compact Kähler manifolds have special cohomological properties not only in the complex framework but also from the symplectic viewpoint: another important result, other than the Hodge decomposition theorem, [Wei58, Théorème IV.3], is the Lefschetz decomposition theorem, [Wei58, Théorème IV.5], which provides a decomposition of the de Rham cohomology in terms of primitive subgroups of the cohomology. Starting from J.-L. Koszul [Kos85] and J.-L. Brylinski [Bry88], several authors studied *symplectic geometry* from the cohomological point of view, see, e.g., [Mat95, Yan96, Cav05, TY12a, TY12b, Lin13]. More precisely, J.-L. Brylinski, aimed by drawing a parallel between the symplectic and the Riemannian cases, proposed in [Bry88] a Hodge theory for compact symplectic manifolds, introducing in particular the notion of symplectically harmonic form; O. Mathieu in [Mat95] and D. Yan in [Yan96] proved that any de Rham cohomology class admits a symplectically harmonic representative if and only if the so-called *Hard Lefschetz Condition* is satisfied. In [TY12a,TY12b], see also [TY11], L.-S. Tseng and S.-T. Yau introduced new cohomologies for symplectic manifolds: among them, in particular, they defined and studied a symplectic counterpart of the Bott-Chern and Aeppli cohomologies, further developing a Hodge theory.

By enlarging the study of the (co)tangent bundle to the study of the direct sum of tangent and cotangent bundles, complex structures and symplectic structures can be framed into a unified context, thanks to the notion of *generalized complex structure*, introduced by N. J. Hitchin in [Hit03] and developed, among others, by M. Gualtieri, [Gua04a, Gua11], and G. R. Cavalcanti, [Cav05], see also [Hit10, Cav07]. The notion of Bott-Chern cohomology, as well as the notion of $\partial\overline{\partial}$-Lemma, can be reformulated also in the generalized complex setting; in particular, one yields the so-called $\mathrm{d}\,\mathrm{d}^{\mathcal{J}}$*-Lemma*, see [Cav05]. Looking at symplectic structures as special

cases of generalized complex structures, the $\mathrm{d}\,\mathrm{d}^{\mathcal{J}}$-Lemma turns out to be just the Hard Lefschetz Condition.

An inequality *à la* Frölicher, characterizing the validity of the Hard Lefschetz Condition, respectively, of the $\mathrm{d}\,\mathrm{d}^{\mathcal{J}}$-Lemma, holds also for symplectic manifolds, respectively for generalized complex manifolds, see Appendix: Cohomological Properties of Generalized Complex Manifolds.

In the second chapter, we consider *nilmanifolds* and, more in general, solvmanifolds. They are defined as compact quotients of connected simply-connected nilpotent, respectively solvable, Lie groups by co-compact discrete subgroups, and they constitute a fruitful and interesting source of examples in non-Kähler geometry. In fact, on the one hand, non-tori nilmanifolds admit no Kähler structure, [BG88, Theorem A], [Has89, Theorem 1, Corollary], and, on the other hand, focusing on *left-invariant* geometric structures on solvmanifolds, one can often reduce their study at the level of the associated Lie algebra; this turns out to hold true, in particular, for the de Rham cohomology of completely-solvable solvmanifolds, [Nom54, Hat60], and for the Dolbeault cohomology of nilmanifolds endowed with certain left-invariant complex structures, [Sak76, CFGU00, CF01, Rol09a, Rol11a], see, e.g., [Con06, Rol11a].

More precisely, on a nilmanifold $X = \Gamma\backslash G$, the inclusion of the subcomplex composed of the G-left-invariant forms on X (which is isomorphic to the complex $(\wedge^\bullet \mathfrak{g}^*,\, \mathrm{d})$, where $\mathfrak{g}$ is the associated Lie algebra) turns out to be a quasi-isomorphism, [Nom54, Theorem 1], that is,

$$i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{R}) \;:=\; H^\bullet\left(\wedge^\bullet \mathfrak{g}^*,\, \mathrm{d}\right) \stackrel{\simeq}{\to} H^\bullet_{dR}(X;\mathbb{R})\;;$$

a similar result holds true also for completely-solvable solvmanifolds, [Hat60, Corollary 4.2], and for the Dolbeault cohomology of nilmanifolds endowed with left-invariant complex structures belonging to certain classes, [Sak76, Theorem 1], [CFGU00, Main Theorem], [CF01, Theorem 2, Remark 4], [Rol09a, Theorem 1.10], [Rol11a, Corollary 3.10].

As a matter of notation, denote by $H^{\bullet,\bullet}_{\sharp}(\mathfrak{g}_{\mathbb{C}})$, for $\sharp \in \left\{\overline{\partial},\, \partial,\, BC,\, A\right\}$, the cohomology of the corresponding subcomplex of G-left-invariant forms on a solvmanifold $X = \Gamma\backslash G$, with Lie algebra $\mathfrak{g}$, endowed with a G-left-invariant complex structure. The following result states a *theorem à la Nomizu* also for the Bott-Chern and Aeppli cohomologies, [Ang11, Theorem 3.7, Theorem 3.8, Theorem 3.9].

Theorem (see Theorems 3.5, 3.6, Remark 3.10, and Theorem 3.7). *Let $X = \Gamma\backslash G$ be a solvmanifold endowed with a G-left-invariant complex structure J, and denote the Lie algebra naturally associated to G by $\mathfrak{g}$. Suppose that the inclusions of the subcomplexes of G-left-invariant forms on X into the corresponding complexes of differential forms on X yield the isomorphisms*

$$i\colon H^\bullet_{dR}(\mathfrak{g};\mathbb{C}) \stackrel{\simeq}{\to} H^\bullet_{dR}(X;\mathbb{C}) \qquad \textit{and} \qquad i\colon H^{\bullet,\bullet}_{\overline{\partial}}(\mathfrak{g}_{\mathbb{C}}) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{\overline{\partial}}(X)\;;$$

in particular, this holds true if one of the following conditions holds:

- X *is holomorphically parallelizable;*
- J *is an Abelian complex structure;*
- J *is a nilpotent complex structure;*
- J *is a rational complex structure;*
- $\mathfrak{g}$ *admits a torus-bundle series compatible with* J *and with the rational structure induced by* Γ*;*
- $\dim_{\mathbb{R}} \mathfrak{g} = 6$ *and* $\mathfrak{g}$ *is not isomorphic to* $\mathfrak{h}_7 := \left(0^3,\ 12,\ 13,\ 23\right)$.

Then also

$$i \colon H^{\bullet,\bullet}_{BC}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{BC}(X) \qquad \textit{and} \qquad i \colon H^{\bullet,\bullet}_{A}\left(\mathfrak{g}_{\mathbb{C}}\right) \stackrel{\simeq}{\to} H^{\bullet,\bullet}_{A}(X)$$

are isomorphisms.

Furthermore, if $\mathcal{C}(\mathfrak{g})$ *denotes the set of* G*-left-invariant complex structures on* X*, then the set*

$$\mathcal{U} \;:=\; \left\{J \in \mathcal{C}(\mathfrak{g}) \;:\; i \colon H^{\bullet,\bullet}_{\sharp_J}\left(\mathfrak{g}_{\mathbb{C}}\right) \;\stackrel{\simeq}{\hookrightarrow}\; H^{\bullet,\bullet}_{\sharp_J}(X)\right\}$$

is open in $\mathcal{C}(\mathfrak{g})$*, for* $\sharp \in \{\partial,\ \overline{\partial},\ BC,\ A\}$.

The above result allows to explicitly compute the Bott-Chern cohomology for the *Iwasawa manifold*

$$\mathbb{I}_3 \;:=\; \mathbb{H}(3;\mathbb{Z}[\mathrm{i}])\backslash\mathbb{H}(3;\mathbb{C})$$

and for its *small deformations*, where

$$\mathbb{H}(3;\mathbb{C}) \;:=\; \left\{\begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{C}) \;:\; z^1,\ z^2,\ z^3 \in \mathbb{C}\right\}$$
$$\text{and} \qquad \mathbb{H}(3;\mathbb{Z}[\mathrm{i}]) \;:=\; \mathbb{H}(3;\mathbb{C}) \cap \mathrm{GL}(3;\mathbb{Z}[\mathrm{i}]) \,.$$

The Iwasawa manifold is one of the simplest example of compact non-Kähler complex manifold: as an example of a holomorphically parallelizable manifold, it has been studied by I. Nakamura, [Nak75], who computed its Kuranishi space and classified the small deformations of $\mathbb{I}_3$ by means of the dimensions of their Dolbeault cohomology groups.

In Sect. 3.2.4, [Ang11, §5.3], we explicitly compute the Bott-Chern cohomology of the small deformations of the Iwasawa manifold, showing that it makes possible to give a finer classification of the small deformations $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\Delta(\mathbf{0},\varepsilon)\subset\mathbb{C}^6}$ of $\mathbb{I}_3$ than the Dolbeault cohomology: more precisely, classes *(ii)* and *(iii)* in I. Nakamura's classification [Nak75, §3] are further subdivided into subclasses *(ii.a)* and *(ii.b)*, respectively *(iii.a)* and *(iii.b)*, according to the value of $\dim_{\mathbb{C}} H^{2,2}_{BC}\left(X_{\mathbf{t}}\right)$.

Left-invariant complex structures on *six-dimensional nilmanifolds* have been classified by M. Ceballos, A. Otal, L. Ugarte, and R. Villacampa in [COUV11]. Hence, in Sect. 3.3, we provide the dimensions of the Bott-Chern cohomology for each of these complex structures, as computed in [AFR12] jointly with M. G. Franzini and F. A. Rossi. In view of [AT13b, Theorem A, Theorem B], such dimensions measure the non-Kählerianity of six-dimensional nilmanifolds.

Enlarging the class of nilmanifolds to *solvmanifolds*, several results concerning de Rham cohomology have been studied by A. Hattori [Hat60], G. D. Mostow [Mos54], D. Guan [Gua07], S. Console and A. M. Fino [CF11], and by H. Kasuya [Kas13a, Kas12a, CFK13]; as for Dolbeault cohomology, results have been obtained by H. Kasuya [Kas13b, Kas12a, Kas11, Kas12d, Kas12b]; as for Bott-Chern cohomology, we have obtained some results on Bott-Chern cohomology in joint work with H. Kasuya, [AK12, AK13a]. In Appendix: Cohomology of Solvmanifolds we summarize some of these results, providing the *Nakamura manifold* as an explicit example.

In the third chapter, we do not require the integrability of the almost-complex structure, and we study cohomological properties of *almost-complex manifolds*, that is, differentiable manifolds endowed with a (possibly non-integrable) almost-complex structure.[1] In this case, the Dolbeault cohomology is not defined. However, following T.-J. Li and W. Zhang, [LZ09], one can consider, for every $p, q \in \mathbb{N}$, the subgroup

$$\begin{aligned} H_J^{(p,q),(q,p)}(X;\mathbb{R}) := &\left\{[\alpha] \in H_{dR}^{p+q}(X;\mathbb{R}) : \alpha \in (\wedge^{p,q}X \oplus \wedge^{q,p}X) \cap \wedge^{p+q}X\right\} \\ \subseteq\; & H_{dR}^{p+q}(X;\mathbb{R}), \end{aligned}$$

and the complex counterpart

$$H_J^{(p,q)}(X;\mathbb{C}) \;:=\; \left\{[\alpha] \in H_{dR}^{p+q}(X;\mathbb{C}) \;:\; \alpha \in \wedge^{p,q}X\right\} \subseteq H_{dR}^{p+q}(X;\mathbb{C}) \;.$$

If X is a compact Kähler manifold, then $H_J^{(p,q)}(X;\mathbb{C}) \simeq H_{\overline{\partial}}^{p,q}(X)$ for every $p, q \in \mathbb{N}$, [DLZ10, Lemma 2.15, Theorem 2.16]; therefore these subgroups can be considered, in a sense, as a generalization of the Dolbeault cohomology groups to the non-Kähler, or to the non-integrable, case.

Two remarks need to be pointed out. Firstly, note that, in general, neither the equality in

$$\sum_{\substack{p+q=k \\ p\leq q}} H_J^{(p,q),(q,p)}(X;\mathbb{R}) \subseteq H_{dR}^{p+q}(X;\mathbb{R}), \quad \text{or} \quad \sum_{p+q=k} H_J^{(p,q)}(X;\mathbb{C}) \subseteq H_{dR}^{p+q}(X;\mathbb{C}),$$

[1] The theory developed by T.-J. Li and W. Zhang for almost-complex structures can actually be restated also in the symplectic, [AT12b], and in the **D**-complex settings, [AR12]. We recall that, in a sense, **D**-*complex geometry* is the "hyperbolic analogue" of complex geometry, and that many connections between it and other theory both in Mathematics and in Physics have been investigated in the last years, see, e.g., [HL83, AMT09, CMMS04, CMMS05, CM09, CFAG96, KMW10, ABDMO05, AS05, Kra10, Ros12a, Ros12b].

holds, nor the sum is direct, nor there are relations between the equality holding and the sum being direct, see, e.g., Proposition 4.1. Hence, one may be interested in studying compact almost-complex manifolds for which one of the above properties holds, at least for a fixed $k \in \mathbb{N}$, see [LZ09, DLZ10, DLZ11, FT10, AT11, AT12a, Zha13, ATZ12, DZ12, HMT11, LT12, DL13, DLZ12]. A remarkable result by T. Drăghici, T.-J. Li, and W. Zhang, [DLZ10, Theorem 2.3], states that every almost-complex structure J on a compact four-dimensional manifold X^4 satisfies the cohomological decomposition

$$H^2_{dR}\left(X^4;\mathbb{R}\right) \;=\; H^{(2,0),(0,2)}_J\left(X^4;\mathbb{R}\right) \oplus H^{(1,1)}_J\left(X^4;\mathbb{R}\right) \;.$$

Secondly, note that $J\lfloor_{\wedge^2 X}$ satisfies $\left(J\lfloor_{\wedge^2 X}\right)^2 = \mathrm{id}_{\wedge^2 X}$; therefore the above subgroups of $H^2_{dR}(X;\mathbb{R})$ can be interpreted as the subgroup represented by J-invariant forms,

$$H^+_J(X) \;:=\; H^{(1,1)}_J(X;\mathbb{R}) \;=\; \left\{[\alpha] \in H^2_{dR}(X;\mathbb{R}) \;:\; J\alpha = \alpha\right\} \;,$$

and the subgroup represented by J-anti-invariant forms,

$$H^-_J(X) \;:=\; H^{(2,0),(0,2)}_J(X;\mathbb{R}) \;=\; \left\{[\alpha] \in H^2_{dR}(X;\mathbb{R}) \;:\; J\alpha = -\alpha\right\} \;.$$

Note also that if g is any Hermitian metric on X whose associated $(1,1)$-form $\omega := g(J\cdot, \cdot\cdot) \in \wedge^{1,1}X \cap \wedge^2 X$ is d-closed (namely, g is an *almost-Kähler* metric on X), then $[\omega] \in H^+_J(X)$.

In fact, T.-J. Li and W. Zhang's interest in studying such subgroups and $\mathcal{C}^\infty$*-pure-and-full* almost-complex structures (that is, almost-complex structures for which the decomposition

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^+_J(X) \oplus H^-_J(X)$$

holds, [LZ09, Definition 2.2, Definition 2.3, Lemma 2.2]) arises in investigating the symplectic cones of an almost-complex manifold, that is, the J*-tamed cone*

$$\begin{aligned}\mathcal{K}^t_J := \{[\omega] \in H^2_{dR}(X;\mathbb{R}) \;:\; & \omega_x\left(v_x,\, J_x v_x\right) > 0 \\ & \text{for every } v_x \in T_x X \setminus \{0\} \text{ and for every } x \in X\}\end{aligned}$$

and the J*-compatible cone*

$$\begin{aligned}\mathcal{K}^c_J := \{[\omega] \in H^2_{dR}(X;\mathbb{R}) \;:\; & \omega_x\left(v_x,\, J_x v_x\right) > 0 \\ & \text{for every } v_x \in T_x X \setminus \{0\} \text{ and for every } x \in X, \text{ and } J\omega = \omega\} \;.\end{aligned}$$

Indeed, they proved in [LZ09, Theorem 1.1] that, given a $\mathcal{C}^\infty$-pure-and-full almost-Kähler structure on a compact manifold X, the J-anti-invariant subgroup $H^-_J(X)$

of $H^2_{dR}(X;\mathbb{R})$ measures the quantitative difference between the J-tamed cone and the J-compatible cone, namely,

$$\mathcal{K}^t_J \;=\; \mathcal{K}^c_J \oplus H^-_J(X)\;.$$

A natural question concerns the qualitative comparison between the tamed cone and the compatible cone: more precisely, one could ask whether, whenever an almost-complex structure J admits a J-tamed symplectic form, there exists also a J-compatible symplectic form. This turns out to be false, in general, for non-integrable almost-complex structures in dimension greater than 4, [MT00, Tom02], see also Theorem 4.6; on the other hand, it is not (yet) known whether, for almost-complex structures on compact four-dimensional manifolds, as asked by S. K. Donaldson, [Don06, Question 2], or for complex structures on compact manifolds of complex dimension greater than or equal to 3, as asked by T.-J. Li and W. Zhang, [LZ09, page 678], and by J. Streets and G. Tian, [ST10, Question 1.7], it holds that $\mathcal{K}^c_J$ is non-empty if and only if $\mathcal{K}^t_J$ is non-empty. We prove in Theorem 4.13 that no counterexample can be found among six-dimensional non-tori nilmanifolds endowed with left-invariant complex structures, [AT11, Theorem 3.3]; note that the same holds true, more in general, for higher dimensional nilmanifolds, as proven by N. Enrietti, A.M. Fino, and L. Vezzoni, [EFV12, Theorem 1.3].

Theorem (see Theorem 4.13). *Let $X = \Gamma\backslash G$ be a six-dimensional nilmanifold endowed with a G-left-invariant complex structure J. If X is not a torus, then there is no J-tamed symplectic structure on X.*

One can study further cones in cohomology, which are related to special metrics, other than Kähler metrics; a key tool is provided by the theory of cone structures on differentiable manifolds developed by D. P. Sullivan, [Sul76]. In order to compare, in particular, the cone associated to *balanced metrics* (that is, Hermitian metrics whose associated $(1,1)$-form is co-closed, [Mic82, Definition 1.4, Theorem 1.6]) and the cone associated to *strongly-Gauduchon metrics* (that is, Hermitian metrics whose associated $(1,1)$-form ω satisfies the condition that $\partial\left(\omega^{\dim_{\mathbb{C}} X-1}\right)$ is $\overline{\partial}$-exact [Pop09, Definition 3.1]), we give the following result, [AT12a, Theorem 2.9], which is the semi-Kähler counterpart of [LZ09, Theorem 1.1]. (We refer to Sect. 4.4.3 for the definitions of the cones $\mathcal{K}b^t_J$ and $\mathcal{K}b^c_J$ on a manifold X endowed with an almost-complex structure J.)

Theorem (see Theorem 4.19). *Let X be a compact $2n$-dimensional manifold endowed with an almost-complex structure J. Assume that $\mathcal{K}b^c_J \neq \varnothing$ (that is, there exists a semi-Kähler structure on X) and that $0 \notin \mathcal{K}b^t_J$. Then*

$$\mathcal{K}b^t_J \cap H^{(n-1,n-1)}_J(X;\mathbb{R}) \;=\; \mathcal{K}b^c_J$$

and

$$\mathcal{K}b^c_J + H^{(n,n-2),(n-2,n)}_J(X;\mathbb{R}) \;\subseteq\; \mathcal{K}b^t_J\;.$$

Moreover, if the equality $H^{2n-2}_{dR}(X;\mathbb{R}) = H^{(n,n-2),(n-2,n)}_J(X;\mathbb{R}) + H^{(n-1,n-1)}_J(X;\mathbb{R})$ *holds, then*

$$\mathcal{K}b^c_J + H^{(n,n-2),(n-2,n)}_J(X;\mathbb{R}) \;=\; \mathcal{K}b^t_J \;.$$

In order to better understand cohomological properties of compact almost-complex manifolds, and in view of the Hodge decomposition theorem for compact Kähler manifolds, it could be interesting to investigate the subgroups $H^{(p,q),(q,p)}_J(X;\mathbb{R})$ for almost-complex manifolds endowed with special structures. For example, we prove the following result, [ATZ12, Proposition 4.1], providing a strong difference between the Kähler case and the almost-Kähler case.

Proposition (see Proposition 4.8). *The differentiable manifold X underlying the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[\mathrm{i}])\backslash\mathbb{H}(3;\mathbb{C})$ admits a non-$\mathcal{C}^\infty$-pure-and-full almost-Kähler structure.*

A further study on almost-Kähler structures $(J,\,\omega,\,g)$ concerns the connections between $\mathcal{C}^\infty$-pure-and-fullness and the *Lefschetz-type property on* 2*-forms* firstly considered by W. Zhang, that is, the property that the Lefschetz operator

$$\omega^{n-2}\wedge\cdot\colon \wedge^2 X \to \wedge^{2n-2} X$$

takes g-harmonic 2-forms to g-harmonic $(2n-2)$-forms, see, e.g., Theorem 4.4; we refer to [ATZ12] for further results.

As a tool to study explicit examples, we provide a Nomizu-type theorem for the subgroups $H^{(p,q),(q,p)}_J(X;\mathbb{R})$ of a completely-solvable solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant almost-complex structure J, [ATZ12, Theorem 5.4], see Proposition 4.2, and Corollary 4.2.

A remarkable result by K. Kodaira and D. C. Spencer states that the Kähler property on compact complex manifolds is stable under *small deformations* of the complex structure, [KS60, Theorem 15]: more precisely, it states that, given a compact complex manifold admitting a Kähler structure, every small deformation still admits a Kähler structure; it can be proven as a consequence of the semi-continuity properties for the dimensions of the cohomology groups of a compact Kähler manifold. Hence, a natural question in non-Kähler geometry is to investigate the (in)stability of weaker properties than being Kähler. As a first result in this direction, L. Alessandrini and G. Bassanelli proved that, given a compact complex manifold, the property of admitting a *balanced metric* (that is, a Hermitian metric whose associated $(1,1)$-form is co-closed) is not stable under small deformations of the complex structure, [AB90, Proposition 4.1]; on the other hand, they proved that the class of balanced manifolds is stable under modifications, [AB96, Corollary 5.7]. Another result in this context is the stability of the property of satisfying the $\partial\overline{\partial}$-Lemma under small deformations of the complex structure, as already recalled, see Corollary 2.2.

Therefore, it is natural to investigate stability properties for the cohomological decomposition by means of the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ on (almost-)complex manifolds $(X,\ J)$. More precisely, we consider the Iwasawa manifold $\mathbb{I}_3 := \mathbb{H}(3;\mathbb{Z}[\mathrm{i}])\backslash\mathbb{H}(3;\mathbb{C})$, showing that the subgroups $H_J^{(p,q),(q,p)}(X;\mathbb{R})$ provide a cohomological decomposition for $\mathbb{I}_3$ but not for some of its small deformations, Theorem 4.8. We prove the following result, [AT11, Theorem 3.2].

Theorem (see Theorem 4.7). *The properties of being $\mathcal{C}^\infty$-pure-and-full is not stable under small deformations of the complex structure.*

More in general, one could try to study directions along which the *curves of almost-complex structures* on a differentiable manifold preserve the property of being $\mathcal{C}^\infty$-pure-and-full. We use a procedure by J. Lee, [Lee04, §1], to construct curves of almost-complex structures through an almost-complex structure J, by means of J-anti-invariant real 2-forms, in order to provide examples, see, e.g., [AT11, Theorem 4.1], see Theorem 4.9.

Another problem in deformation theory is the study of *semi-continuity properties* for the dimensions of the subgroups $H_J^+(X)$ and $H_J^-(X)$. As a consequence of the Hodge theory for compact four-dimensional manifolds, T. Drăghici, T.-J. Li, and W. Zhang proved in [DLZ11, Theorem 2.6] that, given a curve $\{J_t\}_{t\in I\subseteq\mathbb{R}}$ of ($\mathcal{C}^\infty$-pure-and-full) almost-complex structures on a compact four-dimensional manifold X, the functions

$$I \ni t \mapsto \dim_{\mathbb{R}} H_{J_t}^-(X) \in \mathbb{N} \qquad \text{and} \qquad I \ni t \mapsto \dim_{\mathbb{R}} H_{J_t}^+(X) \in \mathbb{N}$$

are, respectively, upper-semi-continuous and lower-semi-continuous. In higher dimension this fails to be true, as we show in explicit examples, [AT12a, Proposition 4.1, Proposition 4.3], see Propositions 4.9 and 4.10. Motivated by such counterexamples, one can study a stronger semi-continuity property on almost-complex manifolds (namely, that, for every d-closed J-invariant real 2-form α, there exists a d-closed J_t-invariant real 2-form $\eta_t = \alpha + \mathrm{o}(1)$, depending real-analytically in t, for $t \in (-\varepsilon,\ \varepsilon)$ with $\varepsilon > 0$ small enough): we give a formal characterization of the curves of almost-complex structures satisfying such a property, [AT12a, Proposition 4.5], see Proposition 4.11, and we provide also a counterexample to such a stronger semi-continuity property, [AT12a, Proposition 4.9], see Proposition 4.12.

The plan of these notes is as follows.

In Chap. 1, we collect the basic notions concerning (almost-)complex, symplectic, and generalized complex structures, we recall the main results on Hodge theory for Kähler manifolds, and we summarize the classical results on deformations of complex structures, on currents and de Rham homology, and on solvmanifolds.

In Chap. 2, we study cohomological aspects of compact complex manifolds, focusing in particular on the study of the Bott-Chern cohomology, [AT13b, AT13a]. By using exact sequences introduced by J. Varouchas, [Var86], we prove a Frölicher-type inequality for the Bott-Chern cohomology, Theorem 2.13, which also provides a characterization of the validity of the $\partial\overline{\partial}$-Lemma in terms of the dimensions of

the Bott-Chern cohomology groups, Theorem 2.14. Finally, we collect some results concerning cohomological aspects of symplectic geometry and, more in general, of generalized complex geometry.

In Chap. 3, we study Bott-Chern cohomology of nilmanifolds, [Ang11, AK12], (see also [AK13a, AK13b]). We prove a result *à la* Nomizu for the Bott-Chern cohomology, showing that, for certain classes of complex structures on nilmanifolds, the Bott-Chern cohomology is completely determined by the associated Lie algebra endowed with the induced linear complex structure, Theorems 3.5, 3.6, and 3.7. As an application, in Sect. 3.2, we explicitly study the Bott-Chern and Aeppli cohomologies of the Iwasawa manifold and of its small deformations. Finally, we summarize some results concerning cohomologies of solvmanifolds.

In Chap. 4, we study cohomological properties of almost-complex manifolds, [AT11, AT12a, ATZ12]. Firstly, in Sect. 4.1, we recall the notion of $\mathcal{C}^\infty$-pure-and-full almost-complex structure, which has been introduced by T.-J. Li and W. Zhang in [LZ09] in order to investigate the relations between the compatible and the tamed symplectic cones on a compact almost-complex manifold and with the aim to throw light on a question by S. K. Donaldson, [Don06, Question 2]. In particular, we are interested in studying when certain subgroups, related to the almost-complex structure, let a splitting of the de Rham cohomology of an almost-complex manifold, and their relations with cones of metric structures. In Sect. 4.2, we focus on $\mathcal{C}^\infty$-pure-and-fullness on several classes of (almost-)complex manifolds, e.g., solvmanifolds endowed with left-invariant almost-complex structures, semi-Kähler manifolds, almost-Kähler manifolds. In Sect. 4.3, we study the behaviour of $\mathcal{C}^\infty$-pure-and-fullness under small deformations of the complex structure and along curves of almost-complex structures, investigating properties of stability, Theorems 4.7 and 4.9, and of semi-continuity for the dimensions of the invariant and anti-invariant subgroups of the de Rham cohomology with respect to the almost-complex structure, Propositions 4.9, 4.10, 4.11, and 4.12. In Sect. 4.4, we consider the cone of semi-Kähler structures on a compact almost-complex manifold and, in particular, by adapting the results by D. P. Sullivan on cone structures, [Sul76], we compare the cones of balanced metrics and of strongly-Gauduchon metrics on a compact complex manifold (see Theorem 4.19).

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Pisa, Italy Daniele Angella

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