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Andrea Braides

Local Minimization, Variational Evolution and Γ -Convergence

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ISBN 978-3-319-01981-9 ISBN 978-3-319-01982-6 (eBook)
DOI 10.1007/978-3-319-01982-6
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2013951143

Mathematics Subject Classification (2010): 49J45, 74Q10, 49J40, 74Q05

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To Krys and Damian

Preface

These notes have been motivated by the interests of the author in variational problems depending on small parameters, for some of which a description based on a global minimization principle does not seem satisfactory. Such problems range from the derivation of physical theories from first principles to numerical problems involving energies with many local minima. Even though an asymptotic description of related global minimization problems can be given in terms of Γ -convergence, the Γ -limit often does not capture the behavior of local minimizers or of gradient flows. This failure is sometimes mentioned as the proof that Γ -convergence is ‘wrong’. It may well be so. The author’s standpoint is that it might nevertheless be a good starting point that may be systematically ‘corrected’.

The author’s program has been to examine the (few) results in the literature and try to connect them with his own work in homogenization and discrete systems, where often the local minimization issues are crucial due to the oscillations of the energies. The directions of research have been to:

- find criteria that ensure the convergence of local minimizers and critical points. In case this does not occur, then modify the Γ -limit into an equivalent Γ -expansion (as introduced by the author and L. Truskinovsky) in order to match this requirement. We note that in this way we ‘correct’ some limit theories, finding (or ‘validating’) other ones present in the literature.
- modify the concept of local minimizer, so that it may be more ‘compatible’ with the process of Γ -limit. One such concept is the small-scale stability of C. Larsen.
- treat evolution problems for energies with many local minima obtained by a time-discrete scheme introducing the notion of ‘minimizing movements along a sequence of functionals’. In this case the minimizing movement of the Γ -limit can always be obtained by a choice of the space- and time-scale, but more interesting behaviors can be obtained at a critical ratio between them. In many cases a ‘critical scale’ can be computed together with an effective motion, from which all other minimizing movements are obtained by scaling. Furthermore the choice of suitable Γ -converging sequences in the scheme above allows to address the issues of long-time behavior and backwards motion.

- examine the general variational evolution results that may be related to these minimizing movements, in particular recent theories of quasistatic motion and gradient flow in metric spaces.

The content of the present notes is taken from a series of lectures which formed a PhD course first given at Sapienza University of Rome from March to May 2012 and subsequently at the University of Pavia from November 2012 to January 2013. Those courses were addressed to an audience of students, some of which with an advanced background (meaning that they were already exposed to the main notions of the Calculus of Variations and of Γ -convergence), and researchers in the field of the Calculus of Variations and of Variational Evolution. This was an advanced course in that it was meant to address some current (or future) research issues rather than to discuss some subject systematically. Part of the notes has also been reworked during a 10-h course at the University of Narvik on October 25–30, 2012.

The reader should bear in mind that the scope of the notes has been to foster discussion on the problems presented rather than construct a general detailed theory (a worthy and very interesting objective, though). Hence, we have focused on highlighting the phenomena and issues linked to the interaction of scales, local minimization and variational evolution, rather than on the details of the Γ -convergence process, or the optimal hypotheses for the definition of gradient flows, and so on, for which we refer to the existing literature.

These notes would not have been written without the personal constant encouragement of Adriana Garroni, who is also responsible for the organization of the PhD course in Rome. I gratefully acknowledge the invitation of Enrico Vitali to give the PhD course in Pavia, his many interesting comments and his delightful hospitality. I greatly profited from the stimulating environments in both departments; special thanks go to all the students who interacted during the course and the final exams. A precious direct contribution has been given by Adriana Garroni for many ideas about the homogenization of damage in Sect. 3.1, by Ulisse Stefanelli, who provided the material for most of Sect. 3.2 by giving a beautiful lecture on the subject during the course at Pavia, and by Luigi Ambrosio for the proofs in Sect. 11.1.1. I also acknowledge the very fruitful discussions with Matteo Focardi, Chris Larsen, Alexander Mielke, Matteo Novaga, Andrey Piatnitski, Giuseppe Savaré and Lev Truskinovsky, which inspired many examples in these notes. I gratefully acknowledge a careful reading of the manuscript by Giovanni Scilla.

Rome, Italy
July 2013

Andrea Braides

Contents

1	Introduction	1
2	Global Minimization	7
2.1	Upper and Lower Bounds	7
2.2	Γ -Convergence	9
2.3	Convergence of Minimum Problems	12
2.4	An Example: Homogenization	14
2.5	Higher-Order Γ -Limits and a Choice Criterion	18
	References	24
3	Parameterized Motion Driven by Global Minimization	25
3.1	A Paradigmatic Example: Damage Models	25
3.1.1	Damage of a Homogeneous Material	26
3.1.2	Homogenization of Damage	30
3.1.3	Dissipations Leading to a Commutability Result	35
3.1.4	Conditions for Commutability	38
3.1.5	Relaxed Evolution	39
3.2	Energetic Solutions for Rate-Independent Evolution	42
3.2.1	Solutions Obtained by Time Discretization	44
3.2.2	Stability	45
3.3	Francfort and Marigo's Variational Theory of Fracture	48
3.3.1	Homogenization of Fracture	50
	References	51
4	Local Minimization as a Selection Criterion	53
4.1	Equivalence by Γ -Convergence	53
4.2	A Selection Criterion	55
4.3	A 'Quantitative' Example: Phase Transitions	56
4.4	A 'Qualitative' Example: Lennard-Jones Atomistic Systems	59
4.5	A Negative Example: Oscillating Perimeters	65
	References	66

5	Convergence of Local Minimizers	67
5.1	Convergence to Isolated Local Minimizers	67
5.2	Two Examples	69
5.3	Generalizations	72
	Reference	78
6	Small-Scale Stability	79
6.1	Larsen's Stable Points	79
6.2	Stable Sequences of Functionals	81
6.3	Stability and Γ -Convergence	82
6.4	Delta-Stable Evolution	87
	References	89
7	Minimizing Movements	91
7.1	An Energy-Driven Implicit-Time Discretization	91
7.2	Time-Dependent Minimizing Movements	98
	References	101
8	Minimizing Movements Along a Sequence of Functionals	103
8.1	Minimizing Movements Along a Sequence	104
8.2	Commutability Along 'Fast-Converging' Sequences	106
8.2.1	Relaxed Evolution	110
8.3	An Example: 'Overdamped Dynamics' of Lennard-Jones Interactions	113
8.4	Homogenization of Minimizing Movements	117
8.4.1	Minimizing Movements for Piecewise-Constant Energies	117
8.4.2	A Heterogeneous Case	120
8.4.3	A Proposal for Some Random Models	124
8.5	Time-Dependent Minimizing Movements	125
8.6	Varying Dissipations: BV-Solutions of Evolution Equations	126
	References	127
9	Geometric Minimizing Movements	129
9.1	Motion by Mean Curvature	129
9.2	A First (Unsuccessful) Generalization	130
9.3	A Variational Approach to Curvature-Driven Motion	132
9.4	Homogenization of Flat Flows	133
9.4.1	Motion by Crystalline Curvature	134
9.5	Homogenization of Oscillating Perimeters	136
9.6	Flat Flow with Oscillating Forcing Term	138
9.6.1	Flat Flow with Forcing Term	138
9.6.2	Homogenization of Forcing Terms	139
	References	143

10	Different Time Scales	145
10.1	Long-Time Behaviour.....	145
10.2	Reversed Time.....	156
	References.....	158
11	Stability Theorems	159
11.1	Stability for Convex Energies.....	159
11.1.1	Convergence Estimates.....	159
11.1.2	Stability Along Sequences of Convex Energies.....	163
11.2	Sandier–Serfaty Theory	166
11.2.1	Convergence of Gradient Flows	167
11.2.2	Convergence of Stable Critical Points	170
	References.....	171
	Index	173