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Invariance Entropy for Deterministic Control Systems

An Introduction

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Para Amar e Helena,
os outros dois cabeças do monstro

Foreword

The topic of mathematical control theory is the exploration of the possibilities and the limitations of changes in dynamical systems due to inputs. Hence connections to the theory of dynamical systems are immanent. In this monograph Christoph Kawan adds a new angle to this connection by bringing to bear concepts and techniques from the global theory of topological and differentiable dynamical systems upon the problem to determine minimal data rates, a very timely subject in control. Here the combination with arguments from nonlinear control seems particularly noteworthy.

This field has developed over the last few years and the present text shows that it has reached a certain maturity. At the same time, I hope that the many open problems will lead to further fruitful investigations.

Augsburg, Germany
June 2013

Fritz Colonius

Preface

This book is supposed to serve as an introduction to the theory of *invariance entropy* which is related to the control task of making a subset in the state space of a control system invariant. Inspired by the seminal work of Nair et al. [85] about notions of entropy measuring the complexity of certain control tasks, and the Bowen–Dinaburg characterizations of topological entropy in metric spaces, Fritz Colonius created the concept of invariance entropy in 2007. At that time, I started to write my Ph.D. thesis under his supervision at the Mathematical Institute of the University of Augsburg and had the pleasure and great opportunity to work on this new topic in the field of information-based control. The text at hand presents the theory obtained in five fruitful years of research in Augsburg and during two research stays in Campinas (Brazil) in August 2010 and in the period from September to November 2011. There I had the chance to work with Luiz San Martin who showed great interest in our research and contributed several important ideas. In this text, the theory as presented so far in the articles [23, 63–65] and in the thesis [62] is also put on a new level of generality. We work with a fairly general definition of control systems which is basically the one that can be found in Sontag's book [102]. Despite the fact that this definition treats discrete- and continuous-time systems simultaneously, the emphasis in this text clearly lies on continuous-time systems given by differential equations. However, where it is no great deal to prove a result also in discrete time and/or in a purely topological setting, we do not hesitate to do so.

The central motivation behind the theory presented in this book comes from the need to deal with communication constraints in digitally networked control systems. Here the assumption of classical control theory that information can be transmitted within control loops instantaneously, lossless, and with arbitrary precision is no longer satisfied. Realistic mathematical models of many important real-world communication and control networks have to take into account general data-rate constraints in the communication channels, time delays, partial loss of information, and variable network topologies. This raises the question about the smallest possible information rate above which a given control task can be solved. Though networked control systems can have a complicated topology, consisting of multiple sensors, controllers, and actuators, a first step towards understanding

the problem of minimal data rates is to analyze the simplest possible network topology, consisting of one controller and one dynamical system connected by a digital channel with a certain rate in bits per unit time. The problem to determine such minimal data rates has been considered for more than 20 years. Early landmarks are the papers by Delchamps [33] who considered quantized information for stabilization and proposed to use statistical methods from ergodic theory and by Wong and Brockett [113] who discussed stabilization of linear systems via coding. From the wealth of literature on this topic there should also be mentioned Tatikonda and Mitter [107], Delvenne [34], Fagnani and Zampieri [41], Liberzon and Hespanha [76], Matveev and Savkin [79], De Persis [36], Savkin [96], and Xie [114]. In these works, mainly linear systems (both deterministic and stochastic) have been considered, and despite different formulations and assumptions, the results therein show that the minimal data rate for stabilization only depends on the unstable open-loop eigenvalues of the system and therefore is independent of the parameters of the coding and control scheme. Nonlinear systems have been considered in [76], where the authors show that global asymptotic stabilization at an equilibrium can be accomplished by using sampled encoded measurements of the state, with a data rate larger than the product of the right-hand side Lipschitz constant and the dimension of the state space. Furthermore, nonlinear systems in feedforward form have been treated in [36], where a hybrid controller is constructed which achieves stabilization at data rates arbitrarily close to zero, in spite of arbitrarily large communication delays. Different control problems for nonlinear systems are treated in [96], namely observability and robustness. Here a systematic approach in terms of a quantity similar to topological entropy of classical dynamical systems leads to a description of the minimal data rate. The research monograph [79] by Matveev and Savkin provides various results concerning state estimation and control of linear and nonlinear systems over channels of limited capacity, including several data rate theorems. In particular, the minimal data rate for observability is related to a notion of topological entropy of the control system. There is much more literature in this field and I apologize to many authors in advance for not mentioning their contributions. A comprehensive and detailed survey with an excellent overview of the literature up to the year 2007 can be found in Nair et al. [86].

The first systematic approach to the problem of minimal data rates for set-invariance and stabilization of (deterministic, nonlinear) control systems was presented in the outstanding paper [85] by Nair, Evans, Mareels, and Moran, which introduced the notion of *topological feedback entropy*. This quantity, which is defined in terms of the open-loop control system, is a measure for the smallest data rate a communication channel connecting a coder and a controller is allowed to have if the system is supposed to solve the control task of rendering a compact subset of the state space invariant. Furthermore, a local version of feedback entropy at an equilibrium is defined which measures the smallest possible data rate for local uniform asymptotic stabilization, and its value is determined by the unstable eigenvalues of the linearization at the corresponding equilibrium.

The definition of topological feedback entropy is similar to the open-cover definition of topological entropy for classical dynamical systems by Adler et al. [1].

The difference, however, is that for topological feedback entropy only such open covers of the given compact set K are considered which can be made invariant in the sense that to each member of the cover a control sequence can be assigned which allows to steer from every state in this open set into the interior of K. Then the entropy of that cover is defined analogously as in the open-cover definition of topological entropy, but the topological feedback entropy of K is defined as the infimum (instead of the supremum) over all such invariant open covers. Looking at this definition, one expects that topological feedback entropy has some properties that are similar to the properties of topological entropy, but that, on the other hand, the similarity is not going too far.

The richness and maturity of the entropy theory in topological and smooth dynamics is based in first line on the variety of alternative definitions which are available next to the open-cover definition. There are the definitions of entropy in terms of separated and spanning sets introduced by Dinaburg [37] and independently by Bowen [10]. Another alternative definition due to Bowen [12] resembles Hausdorff dimension. Arguably the most powerful characterization is given by the variational principle which asserts that the topological entropy is the supremum over the metric entropies with respect to all invariant probability measures of the given system. For topological feedback entropy it was not clear if there was any alternative approach until the concept of invariance entropy, defined as follows, was introduced. For a compact and controlled invariant set Q of a continuous-time control system, one counts for every positive time τ the number of open-loop control functions which are necessary to stay in Q up to time τ from any initial state. Then the exponential growth rate of these minimal numbers as τ tends to infinity defines the entropy. The intuition behind this definition is that a controller which receives a certain amount of information about the state, say n bits, can generate at most 2^n different control functions to steer the system on a finite time interval, and hence the minimal number of control functions needed to accomplish the control task on this time interval is a measure for the necessary amount of information.

The definition of invariance entropy is close in spirit to the Bowen–Dinaburg definition of topological entropy via spanning sets, and because of its conceptual simplicity it allows to draw plenty of more or less obvious consequences immediately. As it turns out, for each one of the properties of topological entropy which are usually considered as elementary the invariance entropy has an analogous property. For linear control systems the analogy goes even far enough that one can use Bowen's formula for the topological entropy of a linear map to give an analogous formula for the invariance entropy.

By its definition invariance entropy measures how fast the number of open-loop control functions grows which are needed to stay in Q for longer and longer times. But next to this obvious meaning it indeed turns out to coincide with topological feedback entropy after the appropriate adaptations to the setting in which the latter is defined, and in this sense invariance entropy is really an alternative way of defining topological feedback entropy.

Before I start to give a description of the book's contents, I provide an overview of the mathematical tools used therein. These mainly come from the classical theory

of dynamical systems, including differential-geometric methods and concepts from ergodic and dimension theory, as well as from geometric control theory. In particular, the applied techniques and results have their origins in the following sources:

- the work on entropy in dynamical systems by Adler et al. [1], Bowen [10, 11], Ito [60], Kolyada and Snoha [70], and many others;
- the work in dimension theory of dynamical systems by Douady and Oesterlé [38], Temam [108], Boichenko, Leonov, and Reitmann [8, 9], Franz [44], Gelfert [49, 50], and Noack [87];
- the work of Nair et al. [85] on topological feedback entropy;
- the control-theoretic work of Colonius and Kliemann (and coauthors) [21,25,26], in particular the theory of control and chain control sets for systems given by differential equations;
- the work of Sontag [100, 101] and Coron [30] on controllability and regularity for control systems given by differential equations;
- the work of Bowen [13], Bowen and Ruelle [14], Young [115], and Liu [78] in ergodic theory of hyperbolic dynamical systems.

The contents of the book are briefly sketched as follows:

The first chapter serves as the introduction of basic control-theoretic notions. As mentioned before, we work with a very general definition of control systems due to Sontag, but we restrict ourselves to time-invariant and complete systems. This definition is given in Sect. 1.1. After that, several particular classes of systems are defined, namely topological, linear, and smooth systems. Section 1.2 establishes the notion of smooth systems given by differential equations which constitute the most important subclass of smooth systems in this book. In Sect. 1.3, the reader is reminded of elementary control-theoretic notions such as orbits, accessibility, and controlled invariant sets. In Sect. 1.4, the control flow of a control-affine system is introduced and its regularity properties are analyzed. Furthermore, control sets and chain control sets are defined and their basic properties are studied. Finally, Sect. 1.5 treats the linearization of a smooth system given by differential equations along a trajectory and the notion of regular control functions.

In Chap. 2, the central notion of invariance entropy for topological time-invariant systems is established and discussed. Also a related notion, named outer invariance entropy, is introduced which in general is only a lower bound for the actual invariance entropy, but in some respect is better behaved. After proving a list of elementary properties in Sects. 2.1 and 2.2, as a first nontrivial example, the invariance entropy of a scalar linear system given by differential equations is computed. Here for the first time a volume growth argument is used to derive a lower bound, which in different variations appears in all of the following chapters and is one of the main ideas in the theory developed in this book. In the last two sections, the relations between invariance entropy and topological feedback entropy as well as minimal data rates are discussed. The central idea here consists in an alternative characterization of invariance entropy in terms of the entropies of the so-called invariant covers of the given controlled invariant set. This leads to the main results, which are the data rate theorem for invariance entropy and a result which

relates both entropies to each other. Additionally, a proof of the data rate theorem for topological feedback entropy is given.

Chapter 3 contains the linear theory. The first main result of this chapter gives a formula for the outer invariance entropy of a linear system. As one expects, under appropriate assumptions, this quantity is given by the sum of the logarithms of the unstable eigenvalues. This corresponds with a multitude of results in the control literature which provide formulas for the minimal data rates for stabilization of linear systems. An important ingredient in the proof of this result is Bowen's formula for the topological entropy of a linear map. The second main result provides an estimate from below for the invariance entropy of an inhomogeneous bilinear system. This lower bound is expressed in terms of the minimal volume growth rate on an invariant subbundle of the control flow of the associated homogeneous system. In continuous time, one can use Selgrade's theorem to choose this subbundle such that the volume growth rate becomes maximal. In this case, the growth rate reduces to the sum of the unstable eigenvalues again if one considers the special case of a linear system.

In Chap. 4, the development of the nonlinear theory begins. In Sect. 4.1, we first prove a result for topological systems, which gives an upper bound for the entropy in terms of a Lipschitz constant and the upper capacitive dimension of the considered subset of the state space. This result is proved in pretty much the same way as the analogous result for topological entropy which has its origins in Kushnirenko [72] and Ito [60] and is nowadays considered as Folklore. The topological result is then adapted to smooth systems on Riemannian manifolds, both in continuous and in discrete time. In the continuous-time case, an appropriate Lipschitz constant can be described in terms of the maximal eigenvalues of the symmetrized covariant derivatives of the right-hand side vector fields. In Sect. 4.2, a general lower bound for a smooth system on a Riemannian manifold with invertible dynamics is given. Here again the volume growth argument is used which leads to an expression in terms of the functional determinants of the transition maps. In the case of a smooth system given by differential equations, the Liouville formula can be used to relate this expression to the divergence of the right-hand side vector fields.

In Chap. 5, the invariance entropy of sets with additional controllability properties is investigated. For simplicity, the main result of this chapter is only proved for smooth systems given by differential equations. This result gives an upper bound for the invariance entropy of a control set in terms of the sum of unstable Lyapunov exponents of a regular periodic trajectory inside the given set. The proof is basically an adaptation of the proof for a result about topological feedback entropy in Nair et al. [85]. Here for the first time classical control-theoretic methods for nonlinear systems enter the scene, and the interplay between the global controllability on the control set and the local controllability along the periodic trajectory is exploited to give the announced result. In general, we are not able to answer the question whether a control set contains regular periodic trajectories. However, for strongly accessible real-analytic systems, Sontag's theorem about universally regular controls yields the existence of plenty of such trajectories, and a more general result of Coron yields such trajectories under considerably weaker assumptions. These trajectories can be used to show that the assumptions of regularity and periodicity in the upper

bound theorem can be weakened under a weak partial hyperbolicity condition. This is carried out in Sect. 5.2.

In Chap. 6, another variant of the volume argument is used to achieve tighter lower bounds for the invariance entropy. The basic idea used here again stems from the classical theory of dynamical systems, more precisely from the theory of escape rates which is closely related to the classical entropy theory and the thermodynamic formalism. Section 6.1 explains this idea in detail. Basically, we use the fact that the invariance entropy is bounded from below by a uniform escape rate from the considered set. This allows to adapt methods from the classical dynamical systems theory to describe the lower bound in terms of volume growth rates and expressions close to topological entropy. Accordingly, instead of control-theoretic assumptions as in Chap. 5, here additional dynamical assumptions have to be imposed on the system, namely, hyperbolicity conditions of weaker or stronger form. The most important ingredients used in this chapter are two volume lemmas for Bowen-balls, the classical one by Bowen and Ruelle [14], in its nonautonomous version proved by Liu [78], and another one by Franz [44] and Gelfert [49, 50].

Finally, Chap. 7 presents examples for the application of the nonlinear theory developed in the preceding three chapters to particular classes of systems. Section 7.1 treats one-dimensional control-affine systems which turn out to be the most nicely behaved class of nonlinear systems. Under appropriate regularity assumptions, here the invariance entropy of a control set can be expressed in terms of the infimum of the Lyapunov spectrum over the control set. If the given system has only one control vector field, this expression can be reformulated in terms of the drift and control vector fields and their derivatives. As an application, a model for the inverse pendulum is studied and the invariance entropy for the region of stabilizability is computed. In Sect. 7.2, we consider the class of nonlinear systems which are uniformly expanding, that is, the systems whose trajectories for a fixed control function exponentially diverge from each other at a rate which is independent of the control function. The main result for this class of systems gives an almost-formula for the invariance entropy of a control set. Section 7.3 again treats inhomogeneous bilinear systems given by differential equations and gives an improvement over the lower estimate of Chap. 3 by using the methods introduced in Chap. 6, and an almost-formula in the case of a control set. Finally, Sect. 7.4 treats projective systems, that is, control-affine systems on projective space which are induced by bilinear systems in Euclidean space. Under the assumption of local accessibility, a complete picture of the maximal regions of controllability of such systems is available. In particular, the control sets with nonempty interior (called *main control sets*) and the chain control sets can be described via the semigroup of the bilinear system and its control flow. Under a hyperbolicity assumption, we are able to provide a formula for the invariance entropy of the open control set in terms of quantities that can be computed directly from the right-hand side of the bilinear system. A thorough analysis of the chain and main control sets shows that these possess a partially hyperbolic structure. Under specific assumptions about the spectrum of the bilinear system, they have a uniformly hyperbolic structure, which allows to apply the main results of Chaps. 5

and 6. However, there are still some unsolved problems that remain if one wants to give a formula for the invariance entropy of all of these sets.

My intention was to keep the book to a large extent self-contained. However, there are some well-known results whose proofs are not given such as Krener's theorem about accessibility, Sontag's theorem about existence of universally regular control functions, Selgrade's theorem, and the existence of finest Morse decompositions. I assume that the reader is familiar with the material taught in standard courses on linear algebra, real analysis, set-theoretic topology, functional analysis, and measure theory. Some supplementary material can be found in the two appendices, mostly without proofs. Appendix A treats some more advanced linear and multilinear algebra as well as basics about differentiable manifolds and Carathéodory differential equations. In Appendix B, some topics related to dynamical systems are covered, in particular, chain recurrence, linear flows on vector bundles, topological entropy, and (sub-)additive cocycles. I hope that the reader who is familiar with the concepts treated in the appendices can skip reading them and may only have to check for the notation introduced there.

Given the subject matter, it is natural that the presented theory is rather incomplete and leaves many questions open. At the end of each chapter, one finds some questions that might be interesting for further research. My hope is that this text is of use in a further development of a systematic analysis of minimal data rate problems in control, and that both mathematicians working in control theory and in dynamical systems will find the problems in this area appealing from an application-oriented and a purely mathematical point of view.

There is a long list of people who I have to thank for their direct or indirect contributions to this piece of work. First of all, there is Fritz who initiated the whole thing, who put and still puts a great effort in pushing it forwards, and never gets tired in developing new ideas and discussing technical details. Then there are the people who showed interest in our work, helped us solving mathematical problems, or contributed new ideas: Tim Bremer, Tomasz Downarowicz, Roberta Fabbri, Ryuichi Fukuoka, Isabell Graf, Lars Grüne, Uwe Helmke, Anne-Marie Hoock, Pei-Dong Liu, Peter Nagel, Girish Nair, Claudio de Persis, Luiz San Martin, Alexandre Santana, Adriano da Silva, Marco Spadini, Ursula Weinhuber, and Fabian Wirth. For proofreading parts of the manuscript I owe thanks to Isabell Graf, Peter Quast, Alexandre Santana, and Helena Soares. Last but in no respect least, there are the people who I owe thanks not so much for their help with mathematical problems, but for their friendship, their hospitality, and their help and support with all kinds of "real problems": My parents, my friends Daniela, Helmut, Ingrid, Isabell, Thomas, and Torben, as well as my Brazilian housemates Amar, Helena, Henrique, Juliana, and Marcos (not to forget Peter, our neighbor). I also acknowledge the financial support of the following grants: DFG grants Co 124/17-1 and 17-2 within DFG Priority Program 1305 and FAPESP grant no. 11/03140-2.

Augsburg, Germany | Christoph Kawan
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Contents

Acronyms

$\emptyset$	The empty set.
A^c	The complement of a subset $A \subset X$, that is, $A^c = X \backslash A$.
$\#A$	For a finite set A, this notation stands for the number of elements in A; if A is infinite, we set $\#A := \infty$.
$A \subset B$	Set inclusion: A is a (not necessarily proper) subset of B.
$A \subsetneq B$	A is a proper subset of B.
Y^X	For sets X and Y, we denote by Y^X the set of all maps $f : X \to Y$.
$f(x) \equiv g(x)$	The maps f and g coincide for all x on their common domain.
$f(\cdot + t)$	If $f : \mathbb{T} \to X$ is a map with $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ and $t \in \mathbb{T}$, this notation is used for the map $s \mapsto f(s+t)$, $\mathbb{T} \to X$.
$f(t\cdot)$	If $f : \mathbb{T} \to X$ is a map with $\mathbb{T} \in \{\mathbb{Z}, \mathbb{R}\}$ and $t \in \mathbb{T}$, this notation is used for the map $s \mapsto f(st)$, $\mathbb{T} \to X$.
$\omega_1\omega_2$	The concatenation of two maps $\omega_1 : [\sigma, \tau) \to U$ and $\omega_2 : [\tau, \mu) \to U$, defined by $$\omega_1\omega_2(t) := \begin{cases} \omega_1(t) \text{ if } t \in [\sigma, \tau), \\ \omega_2(t) \text{ if } t \in [\tau, \mu). \end{cases}$$
ω^μ	If $\omega : [\sigma, \tau) \to U$ is some map, then $\omega^\mu : [\mu + \sigma, \mu + \tau) \to U$ is defined by $\omega^\mu(t) \equiv \omega(t - \mu)$.
$f\|_A$	The restriction of a map $f : X \to Y$ to a subset $A \subset X$.
f^n	The n-th iterate of a map $f : X \to X$, defined inductively by $f^0 := \mathrm{id}_X$ and $f^{n+1} := f \circ f^n$.
$f \times g$	The Cartesian product of two maps $f : X \to X$ and $g : Y \to Y$, that is, $(f \times g)(x, y) = (f(x), g(y))$, $f \times g : X \times Y \to X \times Y$.
id_X	The identity map on a set X, $\mathrm{id}_X(x) \equiv x$.
$\mathbb{1}_A$	The characteristic function of a set A, $$\mathbb{1}_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

$\mathbb{Z}$	The set of all integers.
$\mathbb{N}$	The set of all positive integers.
$\mathbb{Q}$	The set of all rational numbers.
$\mathbb{R}$	The set of all real numbers.
$\mathbb{T}$	This notation simultaneously stands for $\mathbb{Z}$ and $\mathbb{R}$.
$\mathbb{T}_+$	The set of all nonnegative elements of $\mathbb{T}$.
$\mathbb{R}^d$	The d-dimensional Euclidean space $\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ (d copies).
$B(x, \varepsilon)$	In a metric space, $B(x, \varepsilon)$ denotes the open ball of radius ε centered at x.
$N_\varepsilon(A)$	In a metric space, $N_\varepsilon(A)$ denotes the open ε-neighborhood of a set $A \subset X$, that is, the union of all balls $B(x, \varepsilon)$ with $x \in A$.
$\operatorname{dist}(x, A)$	In a metric space (X, ϱ), $\operatorname{dist}(x, A)$ denotes the distance from a point x to a nonempty set $A \subset X$, defined by $\operatorname{dist}(x, A) := \inf_{a \in A} \varrho(x, a)$.
$\operatorname{diam} A$	The diameter of a nonempty subset of a metric space (X, ϱ), $\operatorname{diam} A = \sup_{x,y \in A} \varrho(x, y)$.
$\operatorname{int} A$	The interior of a subset A of a topological space.
$\operatorname{cl} A$	The closure of a subset A of a topological space.
∂A	The boundary of a subset A of a topological space.
$\operatorname{supp} f$	The support of a continuous function $f : X \to \mathbb{R}$, that is, $\operatorname{supp} f := \operatorname{cl}\{x \in X : f(x) \neq 0\}$.
$\lfloor \cdot \rfloor$	For a real number x, we denote by $\lfloor x \rfloor$ the integer part of x, that is, the unique integer such that $x - \lfloor x \rfloor \in [0, 1)$.
$\sigma(T)$	For a linear operator $T : X \to X$, we write $\sigma(T)$ for the spectrum of T, that is, for the set of all eigenvalues.
$\lambda_{\max}(T)$	The maximal eigenvalue of a linear self-adjoint operator T on a Euclidean space X.
$\sigma_i(T)$	The i-th singular value of a linear operator $T : X \to Y$ between Euclidean spaces of the same dimension d, where $\sigma_1(T) \geq \cdots \geq \sigma_d(T)$.
$\mathscr{L}(X, Y)$	For normed vector spaces X and Y, we write $\mathscr{L}(X, Y)$ for the set of all bounded linear maps $T : X \to Y$.
$\\|T\\|$	If $(X, \lvert \cdot \rvert_X)$ and $(Y, \lvert \cdot \rvert_Y)$ are normed vector spaces and $T \in \mathscr{L}(X, Y)$, by $\\|T\\|$ we denote the operator norm of T, that is, $\\|T\\| = \sup_{\lvert x \rvert_X = 1} \lvert Tx \rvert_Y$.
$\ker T$	The kernel of a linear operator $T : X \to Y$, $\ker T = T^{-1}(0)$.
$\operatorname{im} T$	The image of a linear operator $T : X \to Y$, $\operatorname{im} T = T(X)$.
$\det T$	The determinant of a linear operator T between oriented Euclidean spaces of the same dimension.
$\operatorname{tr} T$	The trace of a linear operator T.
T^*	The adjoint operator of a linear operator $T : X \to Y$ between Euclidean spaces of the same dimension.

$\delta_{ij} = \delta_i^j = \delta_j^i$ This notation stands for the Kronecker-Delta, that is,

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

$I = I_d$	The $d \times d$ identity matrix, $I = (\delta_{ij})$.
$\mathrm{GL}(X)$	If X is a vector space, $\mathrm{GL}(X)$ denotes the group of all linear automorphisms of X.
$\mathrm{O}(d)$	This notation stands for the orthogonal group of $\mathbb{R}^d$.
$E^{\perp}$	The orthogonal complement of a subspace E of a Euclidean space.
$x^{\perp}$	The orthogonal complement of the one-dimensional subspace spanned by a nonzero element x of a Euclidean space.
$\angle(x, y)$	The angle between two nonzero vectors x, y in a Euclidean space.
$\mathrm{D}f(x)$	If $f : \mathbb{R}^n \supset D \to \mathbb{R}^m$ is a map defined on an open set D, which is differentiable at x, we write $\mathrm{D}f(x)$ for the Jacobi-matrix of f at x.
$\mathbf{0}$	A boldface zero stands for a constant function $\omega : I \to X$ from an interval I to a vector space X, which is identically 0, that is, $\mathbf{0}(t) = 0 \in X$ for all $t \in I$.
T_pM	The tangent space of a $\mathscr{C}^k$-manifold at $p \in M$.
TM	The tangent bundle of a $\mathscr{C}^k$-manifold.
π_{TM}	The base point projection from TM to M, which sends a tangent vector $v \in T_pM$ to its base point p.
$\mathscr{C}^r(M, N)$	If M and N are $\mathscr{C}^r$-manifolds, then $\mathscr{C}^r(M, N)$ stands for the set of all $\mathscr{C}^r$-maps $f : M \to N$.
$\mathscr{X}^r(M)$	If M is a $\mathscr{C}^{r+1}$-manifold, $\mathscr{X}^r(M)$ stands for the space of all $\mathscr{C}^r$-vector fields on M.
S^d	The unit sphere of dimension d, that is, $\mathrm{S}^d = \{x \in \mathbb{R}^{d+1} : \lvert x\rvert = 1\}$.
$\mathrm{S}(X)$	The unit sphere in a Euclidean space $(X, \langle\cdot,\cdot\rangle)$, $\mathrm{S}(X) = \{x \in X : \langle x, x\rangle = 1\}$.
$\mathbb{P}^d$	The d-dimensional real projective space which is defined as the quotient space of $\mathbb{R}^{d+1}\backslash\{0\}$ by the equivalence relation $x \sim y$ iff $y = \alpha x$ for some nonzero $\alpha \in \mathbb{R}$.
$\mathscr{L}(\gamma)$	The length of a piecewise $\mathscr{C}^1$ or locally absolutely continuous curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval and M a $\mathscr{C}^k$-manifold.
$\exp_p$	If (M, g) is a Riemannian $\mathscr{C}^3$-manifold, $\exp_p$ denotes the Riemannian exponential map at $p \in M$.
$\nabla f(p)$	The covariant derivative of a $\mathscr{C}^1$-vector field f on a Riemannian manifold (M, g) at $p \in M$.
$S\nabla f(p)$	The symmetrized covariant derivative of a $\mathscr{C}^1$-vector field f at p, that is, $S\nabla f(p) = (1/2)(\nabla f(p) + \nabla f(p)^*)$.
$\mathrm{D}X/\mathrm{d}t$	The covariant derivative of a vector field X along a curve.

vol	The Riemannian volume on a Riemannian manifold.
$L^p(I, \mathbb{R}^m)$	The Banach space of all L^p-functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^m$.
ess $\sup_x f(x)$	The essential supremum of a measurable real-valued function $f : X \to \mathbb{R}$, ess $\sup_{x \in X} f(x) = \inf_{N:\, \mu(N)=0} \sup_{x \in X \setminus N} f(x)$.
$\|f\|_{[0,\tau]}$	This notation stands for the L^∞-norm of a function $f \in L^\infty([0,\tau], \mathbb{R}^m)$.
$n(\varepsilon, K)$	The minimal number of ε-balls necessary to cover a totally bounded subset K of a metric space.
$\overline{\dim}_C(X)$	The upper capacitive dimension of a metric space X.
$h_{\mathrm{top},\varrho}(K, f)$	The topological entropy of a uniformly continuous map $f : X \to X$ on a compact subset K of a metric space (X, ϱ).
$h_{\mathrm{top},\varrho}(f)$	The topological entropy of $f : X \to X$ on (X, ϱ).
$\mathscr{V}_1 \oplus \mathscr{V}_2$	The Whitney sum of two subbundles $\mathscr{V}_1$ and $\mathscr{V}_2$ of some vector bundle $\mathscr{W}$.
$\mathrm{rk}\, \mathscr{V}$	The rank of a vector bundle $\mathscr{V}$.