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The Dynamics of Nonlinear Reaction-Diffusion Equations with Small Lévy Noise

Springer

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Preface

Dynamical systems perturbed by small random noise have received a vast attention over the last decades in many areas of science extending from physics through chemistry and biology to climatology. They typically represent a deterministic large scale phenomenon expressed in terms of an ordinary or partial differential equation which inherits the noisy residual of a rapidly fluctuating low intensity perturbation on much smaller scales. Commonly, these systems largely mimic the phenomenon's unperturbed deterministic behavior up to a characteristic time scale. This scale is a function of the intensity of the perturbation, depends essentially on the underlying nature of the noise and, to a minor extent, on the state space geometry of the deterministic system. Beyond that scale the system exhibits noise induced excursions.

If the deterministic system has several stable equilibria to which it converges in generic relaxation times if started in their respective domains of attraction, these excursions lead to transitions between different equilibria starting from small neighborhoods of one of them. If the system is rescaled with its characteristic time scale, the quasi-deterministic waiting time for a transition from an initial equilibrium is of the order of a time unit on an exponential clock. In its characteristic time scale, the complex fluctuating perturbed system therefore behaves asymptotically as a continuous time Markov chain switching between the stable equilibria of the unperturbed system, turning them into *metastable* states.

In the mathematics literature such systems first appeared in the beginning of the 1970s, mainly in the context of large deviations for Gaussian perturbations. For this type of noise, characteristic time scales are of order $\exp(V/\varepsilon^2)$, where ε is the noise intensity, and the quantity V related to the geometry of the deterministic system. The large deviations approach as well as its potential theoretic extension turned out to be very fertile, and large deviation principles describing their metastable behavior have been discovered for large classes of ordinary and partial differential equations.

For dynamical systems with non-Gaussian noise, exit and transition problems have been much less studied. The most interesting non-Gaussian noise is given by the α-stable one, arising in local limit theorems for heavy-tailed random walks. The most prominent example in this class is Cauchy noise, well known to lack

first moments as well as a suitable Cameron–Martin space. Therefore for the study of the metastable behavior of dynamical systems perturbed by it, large deviation techniques may not apply. After an abstract approach via its Markov generator by Godovanchuk in 1979, Imkeller and Pavlyukevich solved the first exit problem for one-dimensional systems and described their metastable behavior in 2006. Their study is crucially based on a skilled distinction between large and small jumps of the noise and the strong Markov property of the system, which allows to compensate for the lack of moments. The precise heuristics behind this approach is explained in detail in Sect. 1.2. In strong contrast to the Gaussian case, the characteristic time scale is of order Q/ε^α, where ε is the noise intensity, α the stability index of the noise, and Q a quantity depending on the geometry of the deterministic system and the Lévy measure.

These lecture notes treat the first exit problem and metastability for a paradigm class of reaction–diffusion equations—the Chafee–Infante equations—perturbed by additive regularly varying noise in the infinite-dimensional space of weakly differentiable functions over an interval. The corresponding principal results are contained in the following theorems. Theorem 5.16 states the convergence of the rescaled first exit times from domains of attraction of equilibria to those of a reduced model in terms of exponential moments on the same probability space. Theorem 7.10 describes metastability for the system in the characteristic time scale. To our knowledge this is the first treatment of this type of problems for stochastic partial differential equations. Also the techniques used in the proofs are new to the field.

The lecture notes address graduate students and researchers in mathematics and natural scientists with a background in partial differential equations and stochastic analysis, who would like to understand in detail the rich and subtle interplay of the deterministic infinite-dimensional dynamics and the jump behavior in terms of the Lévy measure of the random perturbation.

The text is as self-contained as possible with a proof or at least a sketch of it for every proposition in all different areas involved. In particular we give an overview of the literature on the deterministic Chafee–Infante equations. We prove fine estimates on the relaxation time in Chap. 2, which do not exist in the literature so far. In the sequel we give an introduction to stochastic reaction–diffusion equations and establish all properties relevant to our purposes, in particular the existence of a global solution and the strong Markov property in Chap. 3. The mathematical core of the text is presented in Chaps. 4–7. It concludes with an additional chapter in the appendix, where we explain the climate dynamics motivation for our paradigm model.

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Contents

1 Introduction 1
- 1.1 Motivation of the Exit Time Problem from Climate Dynamics 1
- 1.2 Heuristics for the First Exit Times: Noise Decomposition into Small and Large Jumps 4
- 1.3 A Glance at Related Literature 6
- 1.4 Organization of the Book 7

2 The Fine Dynamics of the Chafee–Infante Equation 11
- 2.1 The Classical Dynamics of the Chafee–Infante Equation 11
  - 2.1.1 General Properties of the Solution 11
  - 2.1.2 Domains of Attraction and the Global Attractor 15
- 2.2 Logarithmic Bounds on the Relaxation Time in Reduced Domains 19
  - 2.2.1 Reduced Domains of Attraction 19
  - 2.2.2 Logarithmic Relaxation Times 22
  - 2.2.3 Local Convergence to Stable States 27
  - 2.2.4 Local Repulsion from Unstable States in Reduced Domains 29
  - 2.2.5 Uniform Exit from Small Tubes around Heteroclinic Orbits 38
- 2.3 Proof of Lemma 2.10 39

3 The Stochastic Chafee–Infante Equation 45
- 3.1 Lévy Processes in Hilbert Space and the Noise Decomposition 46
- 3.2 Stochastic Integration in Hilbert Space 51
- 3.3 The Stochastic Convolution with Lévy Noise 54
- 3.4 The Stochastic Chafee–Infante Equation with Lévy Noise 56
- 3.5 The Strong Markov Property 61
- 3.6 Basics on Slowly and Regularly Varying Functions 65

4 The Small Deviation of the Small Noise Solution 69
- 4.1 Uniformly Absorbing Ball for (2.12) 69
- 4.2 Small Deviations of the Small Noise Solution 73

4.3 Small Deviation on Deterministic Time Intervals 74
4.3.1 Small Deviation with Controlled Small Noise Convolution 74
4.3.2 Control of the Small Noise Convolution 79
4.4 Small Deviation before the First Large Jump 83

5 Asymptotic Exit Times 87
5.1 Preparations: Event Estimates and Hypotheses on the Lévy Measure 87
5.1.1 Estimates of Exit Events by Large Jump and Perturbation Events 88
5.1.2 Hypotheses on the Lévy Measure 96
5.2 Asymptotic Exit Times from Reduced Domains of Attraction 100
5.2.1 The Upper Estimate of the Laplace Transform 104
5.2.2 The Lower Estimate of the Laplace Transform 113
5.2.3 Asymptotic Exit Times in Probability 115

6 Asymptotic Transition Times 121
6.1 Asymptotic Times to Enter Different Reduced Domains of Attraction 121
6.2 Transition Times Between Balls Centered in the Stable States 126

7 Localization and Metastability 131
7.1 Hypothesis (H.3) Prevents Trapping Close to the Separatrix 131
7.2 Localization on Subcritical and Critical Time Scales 137
7.3 Metastable Behavior 142

A The Source of Stochastic Models in Conceptual Climate Dynamics ... 151
A.1 A Conceptual Approach to Low-Dimensional Climate Dynamics ... 151
A.1.1 Hasselmann's Unfinished Program 152
A.1.2 Energy Balance Models Perturbed by Noise of Small Intensity 154
A.1.3 The Motivating Phenomenon: Paleoclimatic Warming Events 155

References 159

Notation

Important Constants

- $\alpha \in (0, 2)$, index of the noise, see ν and L
- $\rho \in (\frac{1}{2}, 1)$, see $\frac{1}{\varepsilon^\rho}$, $\varepsilon \in (0, 1)$
- $\Gamma > 0$, large geometric constant
- $\gamma > 0$, appropriately small exponent

The Spaces

- $(L^2(0, 1), |\cdot|)$, Lebesgue space of equivalence classes of square integrable functions on $(0, 1)$ with the usual norm
- $|\cdot|_p$, $p \neq 2$, the norm of the Lebesgue space $L^p(0, 1)$
- $H = H_0^1(0, 1)$, $(H, \|\cdot\|)$, space of weakly differentiable elements of $L^2(0, 1)$ with Dirichlet boundary conditions with $\nabla x \in L^2(0, 1)$ for $x \in H$ and with the norm $\|x\|^2 = \int_0^1 (\nabla x(\zeta))^2 \mathrm{d}\zeta$, $x \in H$
- $B_r(x)$, $x \in H$, $r > 0$ is the ball in H of center x and radius r.
- $(\mathscr{C}_0(0, 1), |\cdot|_\infty)$, space of continuous functions on $[0, 1]$ with $x(0) = x(1) = 0$ with the supremum norm
- $D(\mathbb{R}^+; H)$, space of càdlàg functions on $\mathbb{R}^+ = [0, \infty)$ with values in H

The Deterministic Chafee–Infante Equation

- $(S(t))_{t\geqslant 0}$, heat semigroup on H
- $\lambda > \pi^2$, with $\lambda \neq (k\pi)^2$, $k \in \mathbb{N}$, Chafee–Infante parameter
- $u = (u(t; x))_{\substack{t\geqslant 0\\ x\in H}}$, solution of the deterministic Chafee–Infante equation at time $t \geqslant 0$ with initial value $x \in H$ for fixed parameter λ
- $v_\theta = (v_\theta(t; x))_{\substack{t\geqslant 0\\ x\in H}}$, solution of the deterministic Chafee–Infante equation with nonlinearity $f(\cdot + \theta(t))$ at time $t \geqslant 0$ with initial value $x \in H$ for fixed parameter λ and $\theta \in L^\infty(0, \infty; H)$
- $\phi^\pm$, one of the two stable states $\{\phi^+, \phi^-\}$ of u for fixed λ
- $\mathscr{A}^\lambda$, global attractor of the dynamical system $t \mapsto u(t; \cdot)$ in H for fixed λ

Domains of Attraction

Let $\delta_i > 0, i = 1, 2, 3$, and $\varepsilon, \gamma \in (0, 1)$.

- $D^{\pm}$, domain of attraction of $\phi^{\pm}$ under the flow $t \mapsto u(t; x), x \in H$
- $\mathscr{S} := H \setminus (D^{+} \cup D^{-})$, smooth manifold separating D^{+} and D^{-}, called separatrix
- $D^{\pm}(\delta_1) := \{x \in D^{\pm} \mid \cup_{t\geqslant 0} B_{\delta_1}(u(t; x)) \in D^{\pm}\}$
- $D^{\pm}(\delta_1, \delta_2) := \{x \in D^{\pm} \mid \forall\ \theta \in \mathbb{D}(\mathbb{R}^{+}; H)$ with $\sup_{t\geqslant 0} \|\theta(t)\| \leqslant \delta_2$:
$$\bigcup_{t\geqslant 0} B_{\delta_2}(v_\theta(t; x)) \in D^{\pm}(\delta_1)\}$$
- $\tilde{D}^{\pm}(\varepsilon^{\gamma}) := D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma})$
- $\tilde{D}^{0}(\varepsilon^{\gamma}) := H \setminus (\tilde{D}^{+}(\varepsilon^{\gamma}) \cup \tilde{D}^{-}(\varepsilon^{\gamma}))$
- $\tilde{D}^{\pm 0}(\varepsilon^{\gamma}) := \tilde{D}^{+}(\varepsilon^{\gamma}) \bigcup \tilde{D}^{0}(\varepsilon^{\gamma})$
- $D^{\pm}(\delta_1, \delta_2, \delta_3) := \{x \in D^{\pm} \mid \forall\ \theta \in D(\mathbb{R}^{+}; H)$ with $\sup_{t\geqslant 0} \|\theta(t)\| \leqslant \delta_3$:
$$\cup_{t\geqslant 0} B_{\delta_3}(v_\theta(t; x)) \in D^{\pm}(\delta_1, \delta_2)\}$$
- $\hat{D}^{\pm}(\varepsilon^{\gamma}) := D^{\pm}(\varepsilon^{\gamma}, \varepsilon^{2\gamma}, \varepsilon^{2\gamma})$
- $D^{\pm}(\delta_1, \delta_2, \delta_3, \delta_4) := \{x \in D^{\pm} \mid B_{\delta_4}(x) \in D^{\pm}(\delta_1, \delta_2, \delta_3)\}$
- r^{*}, radius of a ball such that all $v_\theta(\cdot; x)$ enters this ball in a time independent of $x \in H$ and θ, $\sup_{t\geq 0} \|\theta(t)\| \leq 1$, $B_{r^*}(0)$ absorbing set of u
- s_{r^*}, uniform bound from below on the time
- $T_{rec} + \kappa\gamma |\ln \varepsilon|$, upper bound for $u(t; x)$, $x \in D^{\pm}(\varepsilon^{\gamma})$, to enter $B_{(1/2)\varepsilon^{2\gamma}}(\phi^{\pm})$

Shifted Domains of Attraction

Let $\delta_i > 0, i = 1, 2, 3$, and $\varepsilon, \gamma \in (0, 1)$.

- $D_0^{\pm} = D^{\pm} - \phi^{\pm}$
- $D_0^{\pm}(\delta_1) = D^{\pm}(\delta_1) - \phi^{\pm}$
- $D_0^{\pm}(\delta_1, \delta_2, \delta_3, \delta_4) = D^{\pm}(\delta_1, \delta_2, \delta_3, \delta_4) - \phi^{\pm}$
- $\tilde{D}_0^{\pm}(\varepsilon^{\gamma}) = \tilde{D}^{\pm}(\varepsilon^{\gamma}) - \phi^{\pm}$
- $\hat{D}_0^{\pm}(\varepsilon^{\gamma}) = \hat{D}^{\pm}(\varepsilon^{\gamma}) - \phi^{\pm}$
- $\hat{D}^{0}(\varepsilon^{\gamma}) = H \setminus (\hat{D}^{+}(\varepsilon^{\gamma}) \cup \hat{D}^{-}(\varepsilon^{\gamma}))$

The Stochastic Chafee–Infante Equation

- $\varepsilon \in (0, 1)$, noise intensity
- ν, symmetric, regularly varying Lévy measure on $\mathscr{B}(H)$ of index $\alpha \in (0, 2)$
- $L = (L(t))_{t\geqslant 0}$, symmetric pure jump Lévy process in H with Lévy measure ν
- $X^{\varepsilon} = (X^{\varepsilon}(t; x))_{t\geqslant 0}$, solution of the stochastic Chafee–Infante equation driven by $\varepsilon \mathrm{d}L$ at time $t \geqslant 0$ with initial value $x \in H$
- $\Delta_t L = L(t) - L(t-)$, jump of L at time $t > 0$
- $\dfrac{1}{\varepsilon^{\rho}}$, for $\varepsilon, \rho \in (0, 1)$, jump height threshold of L between "small" and "large" jumps
- $\eta^{\varepsilon} = (\eta^{\varepsilon}(t))_{t\geqslant 0}$, compound Poisson process consisting of all jumps of L of height $\|\Delta_t L\| > \frac{1}{\varepsilon^{\rho}}$, called "large" jumps

- $(T_i)_{i\in\mathbb{N}}$, jump times of η^ε
- $t_i = T_i - T_{i-1}, i \in \mathbb{N}$, waiting times between jumps the of η^ε
- $W_i = \Delta_{T_i} L, i \in \mathbb{N}$, i-th jump (increment) of η^ε
- $\xi^\varepsilon = (\xi^\varepsilon(t))_{t\geqslant 0}$, where $\xi^\varepsilon(t) = L(t) - \eta^\varepsilon(t), t \geqslant 0$, called "small" jumps process
- $\xi^* = (\xi^*(t))_{t\geqslant 0}$, where $\xi^*(t) = \int_0^t S(t-s)\mathrm{d}\xi^\varepsilon(s)$, called "small" jumps convolution
- $Y^\varepsilon = (Y^\varepsilon(t;x))_{\substack{t\geqslant 0\\ x\in H}}$, mild solution of the stochastic Chafee–Infante equation driven by $\varepsilon\mathrm{d}\xi^\varepsilon$ at time $t \geqslant 0$ and initial value $x \in H$

Time Scales

Let $\varepsilon > 0$, $\rho \in (\frac{1}{2}, 1)$, $\alpha \in (0,2)$, and write $f_\varepsilon \approx_\varepsilon g_\varepsilon$ for $\lim_{\varepsilon\to 0+} f_\varepsilon/g_\varepsilon = 1$.

- $\lambda^\pm(\varepsilon) = \nu\left(\frac{1}{\varepsilon}(D_0^\pm)^c\right) \approx_\varepsilon \varepsilon^\alpha\ \ell(1/\varepsilon)\ \mu\left((D_0^\pm)^c\right)$, characteristic rate of the first exit time
- $\beta_\varepsilon = \nu\left(\frac{1}{\varepsilon^\rho}B_1^c(0)\right) \approx_\varepsilon \varepsilon^{\alpha\rho}\ \ell(1/\varepsilon^\rho)\ \mu\left(B_1^c(0)\right)$, intensity of η^ε
- $\lambda^0(\varepsilon) = \nu\left(\frac{1}{\varepsilon}B_1^c(0)\right) \approx_\varepsilon \varepsilon^\alpha\ \ell(1/\varepsilon)\ \mu\left(B_1^c(0)\right)$, characteristic rate of metastability
- $\ell : (0,\infty) \to (0,\infty)$, slowly varying function associated with ν
- μ, limit measure of ν on $\mathscr{B}(H)$

Exit Times and Transition Times

Let $\varepsilon, \gamma \in (0,1)$.

- $\tau_x^\pm(\varepsilon)$, first exit time of $X^\varepsilon(\cdot;x), x \in \hat{D}^\pm(\varepsilon^\gamma)$ from the reduced domain of attraction $\tilde{D}^\pm(\varepsilon^\gamma)$
- $\tau_x^{\pm 0}(\varepsilon)$, first exit time of $X^\varepsilon(\cdot;x), x \in \hat{D}^\pm(\varepsilon^\gamma)$ from the enhanced domain of attraction $\hat{D}^{\pm 0}(\varepsilon^\gamma)$
- $\chi_x^\pm(\varepsilon)$, first entrance time of $X^\varepsilon(\cdot;x), x \in \hat{D}^\pm(\varepsilon^\gamma)$ in $B_{\varepsilon^{2\gamma}}(\phi^\mp)$
- $\tau_x^0(\varepsilon)$, first exit time from the neighborhood of the separatrix $\hat{D}^0(\varepsilon^\gamma)$

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