Part II
Non-homogeneous Spaces $(\mathcal{X}, \nu)$

The classical theory of Hardy spaces and singular integrals has been well developed into a large branch of analysis on spaces of homogeneous type in the sense of Coifman and Weiss [18, 19]. Recall that a (quasi-)metric space $(\mathcal{X}, d)$ equipped with a nonnegative measure μ is called a *space of homogeneous type* if $(\mathcal{X}, d, \mu)$ satisfies the *measure doubling condition*: there exists a positive constant $C_{(\mu)}$ such that, for any ball $B(x,r) := \{y \in \mathcal{X} : \ d(x,y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,2r)) \leq C_{(\mu)}\mu(B(x,r)).$$[1]

Typical examples of spaces of homogeneous type include Euclidean spaces, Euclidean spaces with weighted measures satisfying the doubling property, Heisenberg groups, connected and simply connected nilpotent Lie groups and the boundary of an unbounded model polynomial domain in $\mathbb{C}^N$ or, more generally, Carnot–Carathéodory spaces with doubling measures. Since the 1970s, there have been a lot of fruitful results on the theory of Hardy spaces and singular integral operators on spaces of homogeneous type. It is now well known that the space of homogeneous type is a natural setting for the theory of function spaces and singular integrals; see, for example, [1, 18, 19, 24, 47, 99–101].

On the other hand, substantial progress in the study of the theory on function spaces and singular integrals with non-doubling measures disproved the long held belief of the decades of the 1970s and the 1980s that the doubling property of the measures is indispensable in the theory of harmonic analysis.

However, as pointed out by Hytönen in [68], the measures satisfying (0.0.1) do not include the doubling measures as special cases. In [68], Hytönen introduced a new class of metric measure spaces satisfying the so-called geometrically doubling and the upper doubling conditions. This new class of metric measure spaces, which are called *non-homogeneous spaces*, includes both the spaces of homogeneous type and metric spaces with polynomial growth measures as special

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[1] We restrict ourselves to a metric space throughout this book.

cases. Recently, many classical results have been proved still valid if the underlying spaces are replaced by the non-homogeneous spaces (see, for example, [9, 30, 66, 68–70, 89]). It is now also known that the theory of the singular integral operators on non-homogeneous spaces arises naturally in the study of complex and harmonic analysis questions in several complex variables (see [70, 147]). The purpose of this part is to introduce the theory of the Hardy space H^1 and singular integrals in non-homogeneous spaces.

This part consists of two chapters, namely, Chaps. 7 and 8. In Chap. 7, we introduce the non-homogeneous space $(\mathcal{X}, d, \nu)$ and present some basic properties. Based on these properties, we further introduce the atomic Hardy space $H^1(\mathcal{X}, \nu)$ and its dual space, the BMO-type space RBMO $(\mathcal{X}, \nu)$ in this setting, establishing the John–Nirenberg inequality for RBMO $(\mathcal{X}, \nu)$ and some equivalent characterizations of RBMO $(\mathcal{X}, \nu)$ and $H^1(\mathcal{X}, \nu)$, respectively. As applications of Chap. 7, in Chap. 8, we discuss the boundedness of Calderón–Zygmund operators over non-homogeneous spaces $(\mathcal{X}, \nu)$. By establishing the Calderón–Zygmund decomposition, we first show that the Calderón–Zygmund operator T is bounded from $H^1(\mathcal{X}, \nu)$ to $L^1(\mathcal{X}, \nu)$. We then establish the molecular characterization for $H^1(\mathcal{X}, \nu)$ and its variant, $\tilde{H}^1(\mathcal{X}, \nu)$, which is a subspace of $H^1(\mathcal{X}, \nu)$, and obtain the boundedness of T on $\tilde{H}^1(\mathcal{X}, \nu)$. We also prove that the boundedness of T on $L^p(\mathcal{X}, \nu)$, with $p \in (1, \infty)$, is equivalent to its various estimates, and establish some weighted estimates involving the John–Strömberg maximal operators and the John–Strömberg sharp maximal operators, and some weighted norm inequalities for the multilinear Calderón–Zygmund operators. In addition, the boundedness of multilinear commutators of Calderón–Zygmund operators on Orlicz spaces is also presented.

We now make some necessary conventions on notation. As in Part I, throughout this part, we use C, $\tilde{C}$, c and $\tilde{c}$ to denote *positive constants* which are independent of the main parameters, but may change their values at different occurrences. *Constants with subscripts*, such as C_1 and c_1, retain their values at different occurrences throughout this part. Furthermore, $C_{(\rho, \gamma, \ldots)}$ stands for a *positive constant* depending on the parameter ρ, γ, ... Also, the *symbol* $Y \lesssim Z$ means that $Y \le CZ$ for some positive constant C, and $Y \sim Z$ means that $Y \lesssim Z \lesssim Y$.

In this part, unless explicitly pointed out, a *ball* means an open set

$$B := B(x_B, r_B) := \{y \in \mathcal{X} : \ d(x_B, y) < r_B\}$$

with $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$. For any ball $B := B(x_B, r_B)$ and $\varrho \in (0, \infty)$,

$$\varrho B := B(x_B, \varrho r_B).$$

Finally, in this part, we assume that ν is a nonnegative Borel measure on $\mathcal{X}$ and let $\|\nu\| := \nu(\mathcal{X})$. For any set $E \subset \mathcal{X}$, we denote by χ_E the *characteristic function of* E. Moreover, let

$$\operatorname{diam}(\mathcal{X}) := \sup\{d(x, y) : \ x, \ y \in \mathcal{X}\}.$$

For any $f \in L^1_{\mathrm{loc}}(\mathcal{X}, \nu)$ and ball B, $m_B(f)$ denotes the *mean* of f over B, that is,

$$m_B(f) := \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x).$$

For any $p \in [1, \infty]$, in this part, $L^p_b(\mathcal{X}, \nu)$ stands for the *space of functions in* $L^p(\mathcal{X}, \nu)$ *with bounded support* and $L^p_{b,0}(\mathcal{X}, \nu)$ the *space of functions in* $L^p_b(\mathcal{X}, \nu)$ *having integral 0*. We also use $C_b(\mathcal{X})$ to denote the *space of all continuous functions with bounded support*.