Part I
$\mathbb{R}^D$ with Non-doubling Measures μ

The development of the theory of Hardy spaces in the D-dimensional Euclidean space $\mathbb{R}^D$ began with the remarkable paper of Stein and Weiss [122], which was connected closely to the theory of harmonic functions. Later, real variable methods were introduced into this subject by Fefferman and Stein [26], making possible a variety of important D-dimensional results and extensions; see [41, 121, 124]. Moreover, the advent of the methods of the atomic decomposition and molecular decomposition enabled the extension of the theory of Hardy spaces on $\mathbb{R}^D$ to various far more general settings; see [18]. In the development of the theory of Hardy spaces and Calderón–Zygmund operators during the period 1970s–1990s, the only thing that has remained unchanged was the doubling property of the underlying measure, which is a key assumption in the classical theory of harmonic analysis.

Let μ be a nonnegative Radon measure on $\mathbb{R}^D$ which only satisfies the following *polynomial growth condition*, namely, there exist positive constants C_0 and $n \in (0, D]$ such that, for all $x \in \mathbb{R}^D$ and $r \in (0, \infty)$,

$$\mu(B(x,r)) \leq C_0 r^n, \tag{0.0.1}$$

where above and in what follows,

$$B(x,r) := \{y \in \mathbb{R}^D : |x-y| < r\}.$$

Such a measure is not necessary to be doubling. In the recent decades, it was shown that many results on the theory of Hardy spaces and Calderón–Zygmund operators remain valid for non-doubling measures; see [20, 21, 32, 33, 94, 102–106, 131, 132, 134, 141, 151] and references therein. We remark that the analysis in this context, especially, the $T(b)$ theorem and the boundedness of the Cauchy integral on $L^2(\mu)$, plays an essential role in solving the long open Vitushkin's conjecture and Painlevé's problem; see [136, 137, 139] or survey papers [138, 140, 142, 143, 145, 146] for more details. The purpose of this part is to introduce the theory of the Hardy space $H^1(\mu)$ and pay attention to its applications to the study of Calderón–Zygmund operators on $\mathbb{R}^D$ with the measure μ satisfying (0.0.1).

This part consists of six chapters. In Chap. 1, we introduce some basic covering lemmas on $\mathbb{R}^D$ and notions of doubling cubes, and we further establish the Lebesgue differentiation theorem and the Calderón–Zygmund decomposition. In Chap. 2, we introduce a notion of the coefficient for cubes in $\mathbb{R}^D$, which well describes the geometric properties of cubes and is a useful tool in the whole part. Using this notion, we also construct the approximations of the identity on $\mathbb{R}^D$ with μ satisfying (0.0.1). Chapter 3 is devoted to the Hardy space $H^1(\mu)$. We introduce the BMO-type space RBMO (μ), establish the John–Nirenberg inequality for functions in RBMO (μ). We then introduce the atomic Hardy space $H^1(\mu)$, obtain its basic properties, and prove that the dual space of $H^1(\mu)$ is RBMO (μ). A maximal function characterization of $H^1(\mu)$ is also presented. The contents in Chap. 4 involve the study of $h^1(\mu)$ and rbmo (μ), the local versions of $H^1(\mu)$ and RBMO (μ). After presenting some basic properties, corresponding to those of $H^1(\mu)$ and RBMO (μ), of these spaces, we also establish the relations between $H^1(\mu)$ and $h^1(\mu)$ and between RBMO (μ) and rbmo (μ). In addition, we also discuss a BLO-type space RBLO (μ) and its local version rblo (μ) in the present setting. Chapters 5 and 6 are devoted to the study of boundedness of operators. In Chap. 5, we first establish some weighted estimates for the local sharp maximal operators as well as several interpolation results which are useful. Then we investigate the boundedness of the singular integral operators on $L^p(\mu)$ and $H^1(\mu)$, and the boundedness of the maximal singular integral operators and commutators on $L^p(\mu)$ as well as their endpoint estimates. Weighted estimates for (maximal) singular integral operators are also presented. In Chap. 6, we discuss the boundedness of operators associated with approximations of the identity in Chap. 2, on Hardy-type spaces, BMO-type spaces and Morrey-type space, where these operators include Littlewood–Paley operators and maximal operators.

We now make some necessary conventions. Throughout the whole book, C, $\tilde{C}$, c and $\tilde{c}$ stand for *positive constants* which are independent of the main parameters, but they may vary from line to line. *Constants with subscripts*, such as C_0 and A_0, do not change in different occurrences throughout this part. Furthermore, we use $C_{(\rho,\gamma,\ldots)}$ to denote a positive constant depending on the parameter ρ, γ, $\ldots$. Throughout this book, the *symbol* $Y \lesssim Z$ means that there exists a positive constant C such that $Y \le CZ$, and $Y \sim Z$ means that $Y \lesssim Z \lesssim Y$.

In this part, our consideration always takes place in the D-dimensional Euclidean space $\mathbb{R}^D$ (*throughout this book, we use D to denote the dimension of the Euclidean space instead of d because d is used to denote the metric on the non-homogeneous space $\mathcal{X}$ in Part II*). By a *cube* $Q \subset \mathbb{R}^D$, we mean a closed cube whose sides are parallel to the axes and centered at some point of $\operatorname{supp}\mu$, and we denote its *side length* by $\ell(Q)$ and its *center* by z_Q. Given $\lambda \in (0,\infty)$ and any cube Q, λQ stands for the *cube concentric with Q and having side length $\lambda\ell(Q)$*. For any subset $E \subset \mathbb{R}^D$, we denote by χ_E the *characteristic function of E*.

Let μ be a nonnegative Radon measure on $\mathbb{R}^D$, we define $\|\mu\| := \mu(\mathbb{R}^D)$. For any $f \in L^1_{\mathrm{loc}}(\mu)$ and cube Q, $m_Q(f)$ denotes the *mean* of f over cube Q, that is,

$$m_Q(f) := \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

For any $p \in [1, \infty]$, in this part, $L^p_c(\mu)$ stands for the *space of functions in $L^p(\mu)$ with compact support* and $L^p_{c,0}(\mu)$ the *space of functions in $L^p_c(\mu)$ having integral 0*.