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The Hardy Space H^1 with Non-doubling Measures and Their Applications

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Preface

In many problems from analysis, the Hardy space, $H^1(\mathbb{R}^D)$, always appears as a suitable substitution for $L^1(\mathbb{R}^D)$. Thanks to the seminal papers of Charles Fefferman and Elias M. Stein, Ronald R. Coifman and Guido Weiss, Robert H. Latter and other mathematicians, the properties of the Hardy spaces $H^p(\mathbb{R}^D)$ with $p \in (0, 1]$, such as the endpoint spaces as the interpolation spaces, the characterizations in terms of various maximal functions, the atomic and the molecular decompositions, were established in the period 1970s to 1980s. Nowadays, the analysis relating to the Hardy spaces plays an important role in many fields of analysis, such as complex analysis, partial differential equations, functional analysis and geometrical analysis.

On the other hand, one of the most crucial assumptions in the classical harmonic analysis relating to the Hardy space is the doubling condition of the underlying measures. This is because the Vitali covering lemma and the Calderón–Zygmund decomposition lemma—two cornerstones of the classical harmonic analysis—essentially depend on the doubling condition of the underlying measures. For a long time, mathematicians tried to seek a theory about function spaces and the boundedness of operators which does not require the doubling condition on the underlying measures. The motivations for this come from partial differential equations, complex analysis and harmonic analysis itself. One typical example is the singular integral operators considered in an open domain $\Omega \subset \mathbb{R}^D$ with the usual D -dimensional Lebesgue measure, or on a surface with the usual surface area measure instead of the whole space. If the boundary of Ω is a Lipschitz surface, then the problem can be reduced to the related problem in spaces of homogeneous type in the sense of Ronald R. Coifman and Guido Weiss and can be solved by the standard argument. For the domain with extremely singular boundary (or called “wild” boundary), the results for singular integral operators with doubling measures are not suitable anymore. Another famous examples are the so-called Painlevé problem and Vitushkin’s conjecture, in which the non-homogeneous Tb theorem plays a key role. To solve the Painlevé problem, in the 1990s, mathematicians made a great effort to establish the L^2 boundedness for the Cauchy integrals with the one-dimensional Hausdorff measure satisfying some linear growth condition on \mathbb{R} .

Due to the celebrated works concerning the boundedness of the Cauchy integrals with continuous measures, which were given by Guy David, Mark S. Melnikov and Joan Verdera, Xavier Tolsa, Fëdor Nazarov, Sergei Treil and Alexander Volberg, at the end of the last century, mathematicians realized that the doubling condition is superfluous for the boundedness of the Cauchy integral. Since then, the considerable attention in harmonic analysis has been paid to the study of various function spaces and the boundedness of operators on these function spaces over non-homogeneous spaces.

To the best of our knowledge, Alexander Volberg and Xavier Tolsa have already provided two interesting monographs containing the self-contained and unified full proofs of Vitushkin's conjecture and of the semiadditivity of analytic and Lipschitz harmonic capacities (Tolsa's solution of the Painlevé problem), two-weight estimates for the Hilbert transform, as well as some elements of the Calderón–Zygmund theory associated with non-negative Radon measures satisfying some polynomial growth conditions on \mathbb{R}^D , which are also called non-doubling measures on \mathbb{R}^D .

The purpose of this book is to give a detailed survey of the recent progress about the analysis relating to the Hardy space associated with non-doubling measures on \mathbb{R}^D and with non-homogeneous metric measure spaces in the sense of Tuomas Hytönen. The content of the whole book is divided into two parts.

Part I of this book is concerned with the Hardy space $H^1(\mu)$ and its applications on \mathbb{R}^D with non-doubling measures μ , which consists of six chapters. We begin Part I with briefly presenting the history of the development of the real-variable theory of the Hardy space on \mathbb{R}^D and an overview on the main contents of Part I. Then, in Chap. 1, the necessary preliminaries, such as covering lemmas and the Calderón–Zygmund decomposition, are given. In Chap. 2, the approximation of the identity, which is used in the study of operators on functions spaces, is introduced. Chapter 3 is devoted to the Hardy space $H^1(\mu)$. The space $\text{RBMO}(\mu)$, the dual space of $H^1(\mu)$, is also considered in Chap. 3. While in Chap. 4, we study $h^1(\mu)$ and $\text{rbmo}(\mu)$ —the local versions of $H^1(\mu)$ and $\text{RBMO}(\mu)$, respectively. Chapters 5 and 6 are focused on the boundedness on function spaces for the Calderón–Zygmund operators and some classical operators—the Littlewood–Paley maximal operators.

As is well known, the metric space is a natural extension of the Euclidean space, and the analysis relating to metric spaces has its own interest. It is Tuomas Hytönen who overcame some essential difficulties and established a framework for the analysis on non-homogeneous spaces. This new framework turns out to be a geometrically doubling metric space \mathcal{X} with a Borel measure ν satisfying the upper doubling condition. The second part, Part II, of this book is concerned with the analysis relating to the Hardy space $H^1(\mathcal{X}, \nu)$ and the boundedness of Calderón–Zygmund operators over (\mathcal{X}, ν) , which consists of two chapters: Chaps. 7 and 8. Similar to Part I, we begin Part II with briefly presenting the history of the development of the theory of the Hardy space on spaces of homogeneous type and an overview on the main content of Part II. Then, in Chap. 7, we investigate basic properties of this framework, as well as the Hardy space $H^1(\mathcal{X}, \nu)$ and its dual space. While in Chap. 8, the boundedness of Calderón–Zygmund operators,

commutators and some maximal operators in this setting are given. In some sense, the content of Part II is an extension of the results in Part I.

Besides the detailed and self-contained arguments for the main results, after introducing each important notion, we give at least one typical and easily explicable example, which further clarifies the relations between the known and the present notions. At the end of each chapter of this book, there exists a section, called Notes, in which we give the detailed references of the content of this chapter. Also, in Notes, we present more known related results and some unsolved interesting problems, which might be interesting to the reader.

Comparing with the monographs of Alexander Volberg and Xavier Tolsa, only Chaps. 1–3 of this book may partially have some overlaps with the monograph of Xavier Tolsa. The other parts of the present book mainly focus on the results obtained by the authors of this book and their collaborators throughout recent years.

We would like to thank Professor Shanzhen Lu for his consistent encouragement. Dachun Yang thanks Professor Hans Triebel for his consistent encouragement and Professor Yongsheng Han for his fruitful collaborations throughout many years. We also express our appreciations to Professor Xavier Tolsa for his helpful discussions on the subject of Part I. Dachun Yang and Dongyong Yang thank Professor Tuomas Hytönen for fruitful collaborations on the subject of Part II. Last but not least, we wish to thank all our colleagues and collaborators, in particular, we mention several of them: Wengu Chen, Yan Meng, Haibo Lin, Liguang Liu, Wen Yuan, Suile Liu and Xing Fu for their fruitful collaborations throughout these years. Without them, this book would not be presented by this final version.

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Abstract

The theory of the Hardy space is a fundamental tool of Fourier analysis, with applications and connections to complex analysis, partial differential equations, functional analysis and geometrical analysis. It extends to settings where the doubling condition of the underlying measures may fail. Beginning in the 1990s, it was discovered that the doubling condition is superfluous for most results of function spaces and the boundedness of operators. The present book tries to give a gentle introduction to those discoveries, the methods behind them, their consequences and some of their applications. The features of this book are that it provides the detailed and self-contained arguments, many typical and easily explicable examples and interesting unsolved problems.

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