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Green's Kernels and Meso-Scale Approximations in Perforated Domains

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Preface

The book is based on the authors' results on asymptotic approximations of Green's kernels for elliptic boundary value problems in perforated domains. A new feature is the uniformity of the asymptotics with respect to the independent variables. Formal asymptotic approximations are supplied with estimates of the remainder terms. For the case when the number of perforations or inclusions becomes large, a novel method of meso-scale asymptotic approximations is introduced, and uniform asymptotic approximations of Green's kernels as well as solutions of boundary value problems in multiply perforated domains are presented. Such approximations do not require periodicity or other typical constraints attributed to homogenization approximations.

Applications are considered for problems of linear elasticity in planar and three-dimensional domains containing multiple small holes or inclusions. Illustrative computational examples are included to compare asymptotic approximations with accurate finite element numerical simulations, which demonstrate the advantages of the asymptotic method.

This book is addressed to mathematicians, physicists and engineers, as well as research students, interested in asymptotic analysis and numerical computations for solutions to partial differential equations. The required background includes a basic theory of partial differential equations and elements of functional analysis.

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Introduction

There is a wide range of applications in physics and structural mechanics involving domains with singular perturbations of the boundary. Examples include perforated domains and bodies with defects of different types. Accurate direct numerical treatment of such problems is challenging. As alternative means of efficient solution one can use asymptotic approximations. In particular, the multi-scale asymptotic approximations and their justification have been developed for homogenization problems by Marchenko and Khruslov [21], Sánchez-Palencia [39–41], Zhikov [44], Zhikov, Kozlov and Oleinik [43], Allaire [1], Chechkin [4], and Cioranescu and Murat [5].

A comprehensive asymptotic theory of boundary value problems in singularly perturbed domains was developed during the last three decades (see the monographs by Bakhvalov and Panasenko [2], Il'in [15], Kozlov, Maz'ya and Movchan [19], Maz'ya, Nazarov and Plamenevskii [30] and the bibliography therein). This theory includes a general methodology of asymptotic analysis of solutions to boundary value problems, eigenvalues of the corresponding operators and other set functions, such as energy, capacity and stress intensity factors.

In this book, we deal with the analysis of Green's functions and matrices, i.e. kernels of the integral operators representing solutions to elliptic boundary value problems. The exposition is based on the recent work by Maz'ya and Movchan [23–27] and Maz'ya, Movchan and Nieves [28, 29, 32, 33].

The first results on asymptotic approximations of Green's kernels $G_\varepsilon(\mathbf{x}, \mathbf{y})$ for certain classical boundary value problems under small variations of a domain are due to Hadamard [13], who considered regular perturbations of a planar domain with smooth boundary. In connection with our work, it is appropriate to mention that asymptotic approximations in [13] are not uniform with respect to the position of $\mathbf{x}$ and $\mathbf{y}$.

The importance of Green's functions is paramount. Important applications of asymptotic analysis of Green's kernels include extremal problems in the complex function theory in Julia [16], Barnard, Pearce and Campbell [3], shape sensitivity analysis in Fremiot and Sokolowski [11], free boundary problems in Palmerio

and Dervieux [37] and theory of reproducing kernels in Englis et al. [7], and Komatsu [17].

Green's function $G_\varepsilon(\mathbf{x}, \mathbf{y})$ is considered here as the main object for study rather than a tool to generate solutions of specific boundary value problems. Singular perturbations occur while both $\mathbf{x}$ and $\mathbf{y}$ approach the boundary, even in the cases when the boundary itself is smooth. The uniformity of the asymptotic approximations is the principal point of attention. We also show non-trivial links between Green's functions and solutions of boundary value problems for meso-scale structures. Such systems involve a large number of small inclusions, so that a small parameter, the relative size of an inclusion, may compete with a large parameter, represented as an overall number of inclusions.

The main focus of this text is on two topics: (a) asymptotics of Green's functions and tensors for the Laplace and Lamé operators in domains with *singularly perturbed boundaries* and (b) meso-scale asymptotic approximations of physical fields in non-periodic domains with many inclusions. The novel feature of these asymptotic approximations is their *uniformity* with respect to the independent variables.

The book consists of three parts.

The derivation and analysis of the uniform asymptotics of Green's kernels in singularly perturbed domains for the Laplace operator is the main focus of Part I.

To give an impression of such approximations we show the following typical example. Let $G_\varepsilon(\mathbf{x}, \mathbf{y})$ be Green's function of the Dirichlet problem for the operator $-\Delta$ in a two-dimensional domain Ω_ε with a small Jordan inclusion $F_\varepsilon = \{\mathbf{x} : \varepsilon^{-1}\mathbf{x} \in F\}$ (see Fig. 1). We find the asymptotic approximation of G_ε in the form

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) + g(\frac{\mathbf{x}}{\varepsilon}, \frac{\mathbf{y}}{\varepsilon}) - g(\frac{\mathbf{x}}{\varepsilon}, \infty) - g(\infty, \frac{\mathbf{y}}{\varepsilon}) + \frac{1}{2\pi} \log \frac{|\mathbf{x} - \mathbf{y}|}{\varepsilon r_F}$$
$$- \frac{2\pi}{\log(\varepsilon r_F R_\Omega{}^{-1})} \Big(G(\mathbf{x}, 0) + \frac{1}{2\pi} \log \frac{|\mathbf{x}|}{\varepsilon r_F} - g(\frac{\mathbf{x}}{\varepsilon}, \infty) \Big)$$
$$\times \Big(G(0, \mathbf{y}) + \frac{1}{2\pi} \log \frac{|\mathbf{y}|}{\varepsilon r_F} - g(\infty, \frac{\mathbf{y}}{\varepsilon}) \Big) + O(\varepsilon),$$

where G and g are Green's functions of "model" interior and exterior Dirichlet problems in "limit" domains Ω and $\mathbb{R}^2 \setminus F$, independent of ε; r_F and R_Ω are the inner and outer conformal radii of F and Ω, respectively, as defined in Appendix G of Pólya and Szegö [38]. We emphasize that the estimate of the error term in the above asymptotic formula is uniform with respect to $\mathbf{x}$ and $\mathbf{y}$.

Furthermore, we obtain uniform asymptotics of Green's kernels for mixed boundary value problems in domains containing a small hole or a rigid inclusion. We address the Neumann condition on the hole and the Dirichlet condition on the exterior boundary, as well as the Neumann condition on the exterior boundary and Dirichlet condition on the defect. We also derive uniform asymptotics of the Neumann function in the perforated domain. Then, the asymptotic approximations of Green's kernels are constructed in a domain with several small inclusions.

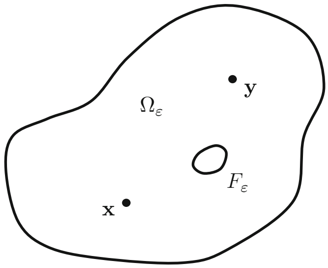

Fig. 1 A domain Ω_ε containing a small hole F_ε

Other examples of asymptotic approximations of Green's functions in singularly perturbed domains include a domain with the singular perturbation of the exterior smooth boundary, a truncated cone and a thin cylindrical body.

Part II is focused on the uniform asymptotic approximations of Green's tensors for linear elasticity in domains with small defects. We obtain uniform asymptotics of Green's tensor in a planar domain and a three-dimensional body containing a small rigid inclusion. This is followed by the construction of uniform asymptotics for Green's tensors in domains with multiple rigid inclusions. Here, instead of the capacitary potential used in approximations of Green's functions for clamped perforated domains in Part I, we introduce the matrix of the *elastic capacity* and study its properties. It will also be shown that this matrix plays an important role in the asymptotic algorithm.

Once the uniform asymptotic approximations for Green's tensor in a domain with multiple small inclusions have been obtained, we consider the asymptotics of Green's tensor in a planar body containing a single small void and furthermore extend this analysis to the case when the body contains several voids. Since the traction conditions are set on the boundary of small defects, we use the dipole fields of linear elasticity in the description of the boundary layer fields.

In Part III, we consider the case when the perforated geometries contain many inclusions or voids of different sizes, as illustrated in Fig. 2, and introduce a novel method of meso-scale asymptotic approximations. First, we deal with asymptotics of solutions to Dirichlet problems for the Poisson equation $-\Delta u = f$ in a three-dimensional body with many perforations. An example of the formal asymptotic representation for the solution of such a boundary value problem is

$$u(\mathbf{x}) \sim v_f(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P^{(j)}(\mathbf{x}) - 4\pi \ \mathrm{cap}(F^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big), \tag{1}$$

where

- v_f is the solution of the same equation in a domain Ω without inclusions
- $P^{(j)}$ is the harmonic capacitary potential of the inclusion $F^{(j)}$

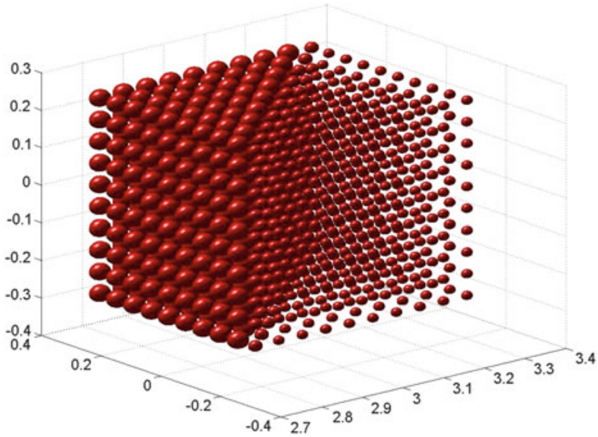

Fig. 2 The non-periodic cloud of 1,000 voids of different sizes. The method of meso-scale asymptotic approximations is applied to obtain a uniform approximation to the solution of the Dirichlet problem in the multiply perforated domain

- $\text{cap}(F^{(j)})$ is the harmonic capacity of $F^{(j)}$
- H is the regular part of Green's function G of Ω

The coefficients C_j satisfy a certain algebraic system, which includes the information about the positions, size and shapes of inclusions.

Furthermore, the text includes meso-scale approximations of Green's function for the Dirichlet problem in a multiply perforated body in $\mathbb{R}^3$:

$$\begin{aligned} G_N(\mathbf{x},\mathbf{y}) = G(\mathbf{x},\mathbf{y}) - \sum_{j=1}^{N} \Bigg\{ & h^{(j)}(\mathbf{x},\mathbf{y}) - P^{(j)}(\mathbf{y}) H(\mathbf{x},\mathbf{O}^{(j)}) \\ & -P^{(j)}(\mathbf{x}) H(\mathbf{O}^{(j)},\mathbf{y}) + 4\pi \,\text{cap}(F^{(j)}) H(\mathbf{x},\mathbf{O}^{(j)}) H(\mathbf{O}^{(j)},\mathbf{y}) \\ & +H(\mathbf{O}^{(j)},\mathbf{O}^{(j)})\, T^{(j)}(\mathbf{x}) T^{(j)}(\mathbf{y}) - \sum_{i=1}^{N} \mathcal{C}_{ij} T^{(i)}(\mathbf{x}) T^{(j)}(\mathbf{y}) \Bigg\} + O(\varepsilon d^{-2}). \end{aligned}$$

Here, d is another small parameter characterizing the minimum distance between each inclusion,

$$T^{(j)}(\mathbf{y}) = P^{(j)}(\mathbf{y}) - 4\pi \,\text{cap}(F^{(j)}) H(\mathbf{O}^{(j)},\mathbf{y}),$$

and again the entries of the matrix $\mathcal{C} = (\mathcal{C}_{ij})_{i,j=1}^{N}$ are solutions of a certain algebraic system containing information about the inclusions.

Moreover, in addition to the meso-scale treatment of Dirichlet problems in domains with many inclusions, we present uniform asymptotic formulae for solutions to mixed boundary value problems in a body with a cloud of many voids,

whose boundaries are subjected to Neumann boundary conditions. Important components of the asymptotic algorithm are the boundary layers near individual voids, whose formal description incorporates the dipole fields characterizing the shape of the voids and their orientation. A model algebraic problem is introduced and solved to evaluate the coefficients in the meso-scale asymptotic approximations. The energy estimates are obtained for the remainder terms.

In particular, for solids containing non-uniform clouds of many spherical voids, the asymptotic approximation takes a form where all boundary layer terms can be written explicitly; this makes such an approximation extremely simple and appealing for numerical implementation in practical problems, where traditional computational approaches like FEM become inefficient.

This book provides an exposition of novel asymptotic approximations, highly efficient for physical problems in multiply perforated domains with non-uniform distribution of defects such as voids or inclusions. The book would be of interest for a mathematician as well as for a physicist or an engineer, who can use the advantage of powerful methods of multi-scale asymptotic approximations in challenging physical problems for composite densely perforated media.