

# Computational Logic and Set Theory

Jacob T. Schwartz • Domenico Cantone •  
Eugenio G. Omodeo

# Computational Logic and Set Theory

Applying Formalized Logic to Analysis

Foreword by Martin Davis

 Springer

Prof. Dr. Jacob T. Schwartz  
(January 9, 1930–March 2, 2009)  
New York University  
New York, NY  
USA

Prof. Domenico Cantone  
Dept. of Mathematics & Computer Science  
University of Catania  
Viale Andrea Doria 6  
95125 Catania  
Italy  
[cantone@dmi.unict.it](mailto:cantone@dmi.unict.it)

Prof. Eugenio G. Omodeo  
Dept. of Mathematics & Computer Science  
University of Trieste  
Via Valerio 12/1  
34127 Trieste,  
Italy  
[comodeo@units.it](mailto:comodeo@units.it)

ISBN 978-0-85729-807-2

e-ISBN 978-0-85729-808-9

DOI 10.1007/978-0-85729-808-9

Springer London Dordrecht Heidelberg New York

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2011934034

© Springer-Verlag London Limited 2011

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms of licenses issued by the Copyright Licensing Agency. Enquiries concerning reproduction outside those terms should be sent to the publishers.

The use of registered names, trademarks, etc., in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant laws and regulations and therefore free for general use.

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

*Cover design:* VTeX UAB, Lithuania

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

*To Diana*

*To Maria Pia and Riccardo*

*To Paola, Pietro, and Sara*



Jacob Theodore “Jack” Schwartz (January 9, 1930–March 2, 2009), courtesy of Diana Robinson Schwartz

# Foreword

Jack Schwartz, the principal, but alas posthumous, author of this book, turned his serious attention to computer science in the mid 1960s. At the time he had already been recognized as a brilliant young mathematician, and the two volumes of the magisterial Dunford–Schwartz *Linear Operators* already in print were widely admired. Jack saw that computers were going to have a revolutionary effect and that the expansions of their use would give rise to many fundamental problems, and he wanted to be part of that. He realized that as software became more complex the question of how its correctness could be ensured would become ever more critical. Moreover he saw formal logic embodied in computer programs as an important part of the answer.

A substantial part of Jack’s research program in computer science derived from his appreciation of the possibility of expressing mathematical discourse in the language of set theory. In much the same way that the seventeenth century work of Descartes and Fermat had shown that propositions of Euclid’s geometry could be regarded as statements in the language of algebra, so the twentieth century contributions of Russell, Zermelo, and von Neumann showed how propositions of the various branches of mathematics could be regarded as statements in the language of set theory. This appreciation led him in three directions:

1. He designed SETL, a general purpose high level programming language based on the language of set theory. The need to achieve acceptable performance from software written in a language that made no concessions to the vagaries of computer architecture led to work on compiler optimization in fruitful collaboration with researchers from IBM.
2. After studying Heinrich Behmann’s algorithm for the decision problem of second order monadic predicate calculus, Jack saw that these methods could be extended to yield algorithms for decidable fragments of set theory. Over a period of decades, working with a group of collaborators almost all from Italy, who were first students at New York University and then became distinguished scientists in their own right, a surprising collection of non-trivial mathematics was found to lie within the scope of such algorithms.

3. Working with some of these same Italian researchers, a computer program was designed and implemented (at least as a prototype) that could verify the correctness of mathematical proofs presented in the language of set theory. Jack proposed to use this verifier to certify the correctness of a substantial body of the fundamentals of mathematical analysis. This was to include proofs of the basic properties of the real and complex number systems defined in set-theoretic terms, the fundamental properties of limits, continuity and the differential and integral calculus, and was to culminate in a proof of the Cauchy Integral Theorem of complex analysis.

The present volume is concerned with this verifier, its use and its context. However this context is to be understood in an extremely broad sense. Some of the work on decidable fragments of set theory is presented in a context that includes other algorithms from various sources for branches of logic as well. The main metamathematical theorems covered in a modern course in mathematical logic are here: the completeness theorem and the two incompleteness theorems of Gödel. Such topics as reflection principles and large cardinals are here as well. Those familiar with Jack Schwartz's mode of thought and with his way of putting his own stamp on a field will have no trouble hearing his voice in this important thought-provoking book.

Martin Davis

Professor Emeritus, Courant Institute, New York University  
Visiting Scholar, University of California, Berkeley

# Preface

In June 2000, the third named author visited New York University and was invited by Jack Schwartz to read what he called the (“common shared”) *scenario*: a wide, carefully assembled sequence of definitions, theorems, and proofs, leading from the bare rudiments of set theory to the beginning of mathematical analysis. Proofs began to be gappy after a few hundred pages, and then totally absent, but the flow of definitions and theorems went on, to culminate in the definition of complex line integral and finally in the celebrated Cauchy integral theorem of complex analysis.

With an implementation appearing all but imminent, Jack had cast a significant piece of mathematics in rigorous formal detail, honestly asking himself whether a computer program could conceivably process and validate every single step. The resulting large-scale proof scenario was meant—in Jack’s own words—“to serve as an essential part of the feasibility study that must precede the development of any ambitious proof-checker”. Eventually, it would also serve as a testing-bench for the concrete implementation of the proof-checker.

The conception of this book on computational logic began then. According to our initial plans, the book would have described the structure of a proof verifier rooted in set theory and would also have surveyed a twenty-year long stream of results on decidable fragments of set theory.

One year later, the second named author visited New York in his turn; at his request Jack advanced the implementation work, speedily bringing into existence the proof-checker Referee, also known as Ref, or as *ÆtnaNova*. This is still a prototype, but it is reliable and fast enough to give us the possibility of debugging our proof scenarios. Some, though not all, of the content of this book is thus related to concrete experience, which we are now pleased to share with our readers.

A very large proof scenario is available today as a  $\text{\LaTeX}$ -generated PDF-file. But given its size (over a thousand pages), it seems appropriate to publish it on the web (and eventually as a CD) rather than to print it. As for the proof verifier, it is usable on the web, but it depends on a SETL2 implementation. Since there is hardly anyone maintaining the SETL system today, Jack undertook with us a re-implementation of the proof verifier in a currently more popular language. But this will take some time; it should not be permitted to delay the publication of this book.



This is a posthumous publication, as its principal author passed away on March 2, 2009. In spite of his long illness, until the end of his life, Jack showed unbelievable resources of energy in the preparation of this book, in implementing and debugging Ref, and in drafting and writing the scenario. With the inspirer of this work gone, the book may not have achieved the degree of perfection that had been Jack Schwartz's goal; nevertheless we believe that the material he left behind will attract many readers and that its publication will be an appropriate tribute to a distinguished scientist.

### ***A Word on the Audience for Whom This Book Is Intended***

Any technical book must, by emphasizing certain details and leaving others unspoken, make definite assumptions about the prior knowledge of the reader.

This book assumes that the reader has a good knowledge of standard programming techniques, particularly of string manipulation and parsing, and also a general familiarity with those parts of mathematics that are analyzed in detail in the main series of definitions and proof scenarios to which much of the book is devoted.

On the other hand, little knowledge of formal logic is assumed. For this reason we try to present what is needed from logic in a reasonably self-contained way, emphasizing concepts likely to be important in continuations of the work begun here, rather than technicalities. Foundational issues, for example consideration of the strength or necessity of axioms, or the precise relationship of our formalism to other weaker or stronger formalisms studied in the literature, are neglected.

Because we expect our readers to be programmers of some sophistication, syntactic details of the kind that often appear early in books on logic are underplayed, and we repeatedly assume that anything programmable with relative ease can be taken as routine, and that the properties of such programmable operations can be proved when necessary to some theoretical discussion. This reflects our feeling that understanding develops top-down, focusing on details only as these become necessary.

We believe that too much detail is more likely to impede than to promote understanding. Who reads, or would want to read, the Whitehead–Russell *Principia*, or could testify that its hundreds of formula-filled pages are without error? But since we ask this question, why do we include hundreds of formula-filled pages in this book, which would not exist without the pioneering work of Whitehead and Russell? The reason lies in the fact that our formal proof text is, to a large extent, computer-checked. Though relatively useless to the human reader unless their correctness can be verified mechanically, long lists of formulae become useful once such verification becomes possible.

## *Content of This Book*

Chapter 1 gives rapid overviews of the authors' approach to automated proof verification and of the large-scale formalized proof scenario whose development has accompanied the writing of this book.

Chapter 2 prepares for an extensive account of our proof verifier *ÆtnaNova*, by surveying three traditional branches of logic: propositional calculus, first-order predicate calculus, set theory. Completeness proofs are provided for the first two of these deductive systems; the much-debated issue of the consistency of Zermelo–Fraenkel set theory and of some of its proposed extensions is highlighted.

Chapter 3 provides an extended survey of inference mechanisms. Some of these belong to the initial endowment of *ÆtnaNova*, others are candidates for inclusion in that endowment should our proof verifier be re-implemented or enhanced. In some cases efficiency considerations show that an inference mechanism cannot be applied at its fullest; notwithstanding we present it because of the mathematical insight it provides. Two classics of the automated deduction field, Robinson's resolution principle and the Knuth–Bendix equational method, are also surveyed in this chapter.

Chapter 4 describes our verifier and its underlying design in more detail. In Chapter 5 we expand a broad survey of main definitions and theorems, showing the salient steps of a formalized proof scenario leading toward the (as yet) unachieved goal of proving the Cauchy integral theorem.

In Chapter 6, for completeness sake and to enjoy the intellectual insight that these results provide, we derive several of the main classical results on undecidability and unsolvability; in particular, Chaitin's theorem and the two celebrated Gödel's incompleteness theorems.

To convey the character of a scenario verifiable by means of our *ÆtnaNova* system, we conclude with Chapter 7 showing formalized proofs of many facts about ordinals, of various properties of the transitive closure operation, of finite and transfinite induction principles, and of Zorn's lemma.

## *Acknowledgements*

We are grateful to Martin Davis, to Alfredo Ferro, and to Alberto Policriti for encouraging, through decades of scientific interaction with the authors, the maturation of many ideas in this book; to Salvatore Paxia, who enabled us to keep the *ÆtnaNova* system alive; to Alexandru Ioan Tomescu, who contributed to the development of proof scenarios. Various results reported in this book stem from joint work with Gianluca Cincotti, Piero Ursino, and Calogero Zarba; Emanuele Giaquinta gave us precious advice concerning  $\text{\LaTeX}$ .

Diana Robinson Schwartz gave us major support throughout the preparation of this book.

This research was partially funded by MIUR/PRIN project 2006/2007 “*Large-scale development of certified mathematical proofs*” No. 2006012773, and by INdAM/GNCS (Istituto Nazionale di Alta Matematica “F. Severi”, Gruppo Nazionale per il Calcolo Scientifico).

Catania, Italy  
Trieste, Italy

Domenico Cantone  
Eugenio G. Omodeo

# Contents

<b>1</b>	<b>Introduction</b> . . . . .	1
1.1	Loomings . . . . .	1
1.1.1	The Special Nature of Mathematical Reasoning Within Human Reason in General . . . . .	5
1.2	Proof Verifiers . . . . .	6
1.3	Informal Introduction to the Formalism in which We Will Work . .	7
1.3.1	(A) Immediate Deduction . . . . .	7
1.3.2	(B) Proof by ‘Supposition’ and ‘Discharge’ (‘Natural Deduction’) . . . . .	8
1.3.3	(C) Use of Definitions . . . . .	8
1.4	More About Our Formalism . . . . .	10
1.4.1	Propositional and Predicate Calculus . . . . .	10
1.4.2	Set Theory: The Third Main Ingredient of Our Formalism .	13
1.5	An Informal Overview of the Sequence of Formal Set-Theoretic Proofs to Be Given Later . . . . .	25
1.5.1	Basic Elementary Results . . . . .	25
1.5.2	Ordinals . . . . .	25
1.5.3	Well Ordering: The Principle of Transfinite Enumerability .	27
1.5.4	Cardinal Numbers . . . . .	29
1.5.5	Survey of the Major Sequence of Definitions and Proofs Considered in This Text . . . . .	32
<b>2</b>	<b>Propositional- and Predicate-Calculus Preliminaries</b> . . . . .	37
2.1	The Propositional Calculus . . . . .	37
2.2	The Predicate Calculus . . . . .	44
2.2.1	Proof Rules of the Predicate Calculus . . . . .	51
2.2.2	The Gödel Completeness Theorem . . . . .	52
2.2.3	Working with Universally Valid Predicate Formulae. A Few Simple Examples of Predicate Proof . . . . .	54
2.2.4	The Prenex Normal Form of Predicate Formulae . . . . .	60
2.2.5	The Deduction Theorem . . . . .	60

- 2.2.6 Definitions in Predicate Calculus; the Notion of ‘Conservative Extension’ . . . . . 62
- 2.2.7 Proof of the Gödel Completeness Theorem . . . . . 65
- 2.3 Predicate Calculus with Equality as a Built-in . . . . . 75
- 2.4 Set Theory as an Axiomatic Extension of Predicate Calculus . . . . . 77
  - 2.4.1 Zermelo–Fraenkel Theory with the Axiom of Choice . . . . . 77
  - 2.4.2 Concerning the Consistency of ZFC and Various Interesting Extensions of It . . . . . 79
- References . . . . . 91
- 3 A Survey of Inference Mechanisms . . . . . 93**
  - 3.1 The Davis–Putnam Propositional Decision Algorithm . . . . . 93
    - 3.1.1 Horn Formulae and Sets of Formulae . . . . . 95
    - 3.1.2 Reducing Collections of Propositional Formulae to Collections of Standardized Disjunctions . . . . . 96
  - 3.2 Elementary Boolean Theory of Sets . . . . . 97
    - 3.2.1 Elementary Boolean Theory of Sets, Plus the Predicates ‘Finite’ and ‘Countable’ . . . . . 100
    - 3.2.2 Elementary Boolean Operators on Sets, with the Cardinality Operator and Additive Arithmetic on Integers . . 102
    - 3.2.3 Quantified Predicate Formulae Involving Predicates of One Argument Only . . . . . 104
  - 3.3 MLSS: Multilevel Syllogistic with Singletons . . . . . 109
  - 3.4 MLSS Plus the Predicates ‘Finite’ and ‘Countable’ . . . . . 113
  - 3.5 The Tableau Method . . . . . 115
  - 3.6 Elementary Booleans Plus Map Primitives . . . . . 120
  - 3.7 Various Commonly Occurring Decidable Extensions of MLSS . . . . 122
    - 3.7.1 Extension Conditions in the Other Cases Listed Above . . . . 126
    - 3.7.2 The Case of Mutually Inverse Functions . . . . . 129
  - 3.8 More Examples of Decidable Sublanguages . . . . . 131
    - 3.8.1 Presburger’s Decidable Quantified Language of Additive Arithmetic . . . . . 131
    - 3.8.2 A Decidable Quantified Theory Involving Ordinals . . . . . 134
    - 3.8.3 A Language of Additive Infinite Cardinal Arithmetic . . . . . 148
    - 3.8.4 Behmann’s Quantified Language of Elementary Set-Theoretic Formulae . . . . . 151
  - 3.9 A Decision Algorithm for the Theory of Totally Ordered Sets . . . . 157
  - 3.10 A Decision Algorithm for Ordered Abelian Groups . . . . . 159
  - 3.11 A Fragment of Analysis: Theory of Reals and Single-Valued Continuous Functions with Predicates ‘Monotone’, ‘Convex’, ‘Concave’, Real Addition, and Comparison . . . . . 165
    - 3.11.1 Syntax of  $\text{RMCF}^+$  . . . . . 165
    - 3.11.2 Semantics of  $\text{RMCF}^+$  . . . . . 166
    - 3.11.3 Preparing a Set of  $\text{RMCF}^+$  Statements for Satisfiability Testing . . . . . 169
  - 3.12 The Resolution Method for Pure Predicate-Calculus Proving . . . . 177

- 3.12.1 Resolution in the Propositional Calculus . . . . . 179
- 3.12.2 Resolution and Syntactic Unification in the Predicate  
Calculus . . . . . 180
- 3.13 Universally Quantified Predicate Sentences Involving Function  
Symbols of One Argument Only . . . . . 190
- 3.14 The Knuth–Bendix Equational Method . . . . . 193
  - 3.14.1 Overview of the Method . . . . . 193
  - 3.14.2 Details . . . . . 195
  - 3.14.3 Testing Completeness by Superposition of Reductions:  
The Knuth–Bendix Completion Process . . . . . 199
  - 3.14.4 More Details . . . . . 200
  - 3.14.5 Examples of the Knuth–Bendix Procedure . . . . . 200
  - References . . . . . 202
- 4 More on the Structure of the Verifier System . . . . . 205**
  - 4.1 Introduction to the General Syntax and Overall Structure of Proofs 205
    - 4.1.1 The Syntax of Proofs . . . . . 205
    - 4.1.2 The ELEM Primitive and ‘Blobbing’ . . . . . 208
    - 4.1.3 The Suppose\_not, QED, Suppose, Discharge Primitives . . 210
    - 4.1.4 THEORY Application . . . . . 211
    - 4.1.5 Context of an Inference Step . . . . . 213
  - 4.2 The Syntax and Semantics of Definitions . . . . . 215
  - 4.3 Other Techniques Used in the Verifier as Implemented . . . . . 218
    - 4.3.1 Supplementary Proof Mechanisms for the ELEM Rule . . . 218
    - 4.3.2 Limited Predicate Proof . . . . . 219
    - 4.3.3 Proof by Equality . . . . . 224
    - 4.3.4 Proof by Monotonicity . . . . . 224
    - 4.3.5 Algebraic Deduction . . . . . 227
    - 4.3.6 Proof by Closure . . . . . 229
    - 4.3.7 The Behind-the-Scenes Activity of Proof by Structure . . . 230
    - 4.3.8 ‘Blobbing’ More General Formulae Down to a Specified  
Decidable or Semi-decidable Sublanguage of Set Theory . . 235
    - 4.3.9 Accelerated Instantiation of Quantifiers and Set Formers . . 236
    - 4.3.10 Computation with Hereditarily Finite Sets . . . . . 238
  - 4.4 Dividing Long Proof Verifications into Multiple Separate ‘Sessions’ 253  
References . . . . . 255
- 5 A Closer Examination of the Sequence of Definitions and Theorems  
Presented in this Book . . . . . 257**
  - 5.1 Basic Operations of Set Theory and the Theory of Ordinals . . . . 258
    - 5.1.1 Pairs, Set Formers, and Maps . . . . . 258
    - 5.1.2 Transfinite Induction . . . . . 260
    - 5.1.3 Ordinals . . . . . 260
    - 5.1.4 The Ordinal Enumerability Theorem . . . . . 261
  - 5.2 Elementary Laws on Map Constructs . . . . . 262
  - 5.3 Cardinality of a Set; Cardinal Numbers . . . . . 268

- 5.3.1 Finiteness . . . . . 270
- 5.4 The Set of All Integers, Basic Arithmetic of Integers and Cardinals 273
- 5.5 The Cardinal Product Theorem . . . . . 279
- 5.6 The Signed Integers . . . . . 281
- 5.7 Induction Principles for Ordinals . . . . . 285
  - 5.7.1 Mathematical Induction for Integers . . . . . 287
- 5.8 Equivalence Relationships and Classes; the General Summation Operator; Recursion . . . . . 287
- 5.9 Formal Fractions and Rational Numbers . . . . . 289
- 5.10 Real Numbers . . . . . 295
- 5.11 Complex Numbers . . . . . 300
- 5.12 Functions of Real and Complex Variables . . . . . 302
  - References . . . . . 311
- 6 Undecidability and Unsolvability . . . . . 313**
  - 6.1 Chaitin’s Theorem . . . . . 313
    - 6.1.1 Undecidability Results Derivable from Chaitin’s Theorem . 315
  - 6.2 The Two Gödel Theorems . . . . . 319
    - 6.2.1 Programming Considerations . . . . . 320
    - 6.2.2 Programming and Proof; ‘Mirroring’ Programmable Set-Theoretic Functions . . . . . 323
    - 6.2.3 Additional Comments on the Legitimacy of Recursive Definitions . . . . . 328
    - 6.2.4 Properties of Integers . . . . . 329
    - 6.2.5 A Final Remark on Proof and Computation . . . . . 336
    - 6.2.6 A Technical Adjustment . . . . . 336
    - 6.2.7 The ‘Provability’ Predicate  $Pr(s)$  . . . . . 337
    - 6.2.8 Proof Visibility Lemma . . . . . 339
    - 6.2.9 Gödel’s Trick Sentence . . . . . 343
    - 6.2.10 Rosser’s Variant of Gödel’s Trick Sentence . . . . . 344
    - 6.2.11 Proof of Rosser’s Variant of Gödel’s First Theorem . . . . 345
    - 6.2.12 Proof of Gödel’s Second Theorem . . . . . 346
  - 6.3 Axioms of Reflection . . . . . 346
    - 6.3.1 Statement of the Axioms of Reflection . . . . . 359
  - 6.4 A Digression Concerning Foundations . . . . . 367
    - References . . . . . 371
- 7 A Self-contained Beginning for Ref’s Main Proof Scenario . . . . . 373**
  - 7.1 Axioms of Set Theory . . . . . 373
  - 7.2 Pairs and Maps . . . . . 374
  - 7.3 From Reachability to Transfinite Induction . . . . . 378
    - 7.3.1 Reachability in a Big Graph . . . . . 378
    - 7.3.2 Full Sets and Ordinals . . . . . 386
    - 7.3.3 The Transitive Closure Operation . . . . . 390
    - 7.3.4 A Basic Form of the Principle of Transfinite Induction . . . 392
    - 7.3.5 Some Basic Facts on Ordinal Numbers . . . . . 393

Contents	xvii
7.4 Zorn's Lemma . . . . .	398
7.5 Finiteness . . . . .	405
References . . . . .	409
<b>Index</b> . . . . .	<b>411</b>