Computational Logic and Set Theory

Jacob T. Schwartz • Domenico Cantone •
Eugenio G. Omodeo

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Applying Formalized Logic to Analysis

Foreword by Martin Davis

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Prof. Dr. Jacob T. Schwartz
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New York University
New York, NY
USA

Prof. Domenico Cantone
Dept. of Mathematics & Computer Science
University of Catania
Viale Andrea Doria 6
95125 Catania
Italy
cantone@dmi.unict.it

Prof. Eugenio G. Omodeo
Dept. of Mathematics & Computer Science
University of Trieste
Via Valerio 12/1
34127 Trieste,
Italy
eomodeo@units.it

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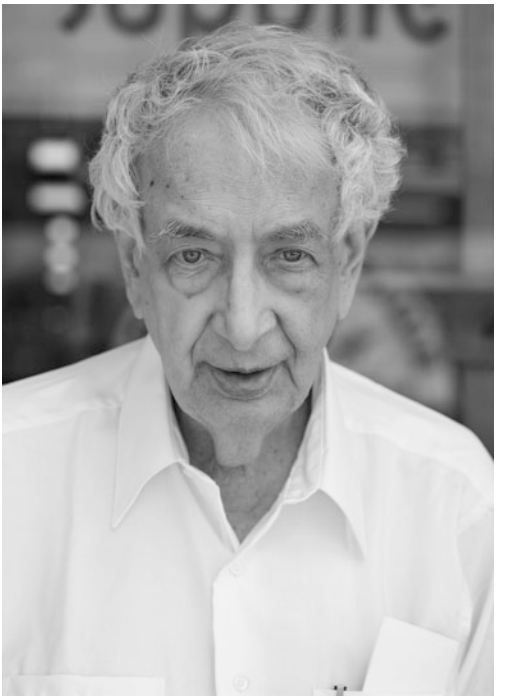
To Diana

To Maria Pia and Riccardo

To Paola, Pietro, and Sara



Jacob Theodore “Jack” Schwartz (January 9, 1930–March 2, 2009), courtesy of Diana Robinson Schwartz

Foreword

Jack Schwartz, the principal, but alas posthumous, author of this book, turned his serious attention to computer science in the mid 1960s. At the time he had already been recognized as a brilliant young mathematician, and the two volumes of the magisterial Dunford–Schwartz *Linear Operators* already in print were widely admired. Jack saw that computers were going to have a revolutionary effect and that the expansions of their use would give rise to many fundamental problems, and he wanted to be part of that. He realized that as software became more complex the question of how its correctness could be ensured would become ever more critical. Moreover he saw formal logic embodied in computer programs as an important part of the answer.

A substantial part of Jack's research program in computer science derived from his appreciation of the possibility of expressing mathematical discourse in the language of set theory. In much the same way that the seventeenth century work of Descartes and Fermat had shown that propositions of Euclid's geometry could be regarded as statements in the language of algebra, so the twentieth century contributions of Russell, Zermelo, and von Neumann showed how propositions of the various branches of mathematics could be regarded as statements in the language of set theory. This appreciation led him in three directions:

1. He designed SETL, a general purpose high level programming language based on the language of set theory. The need to achieve acceptable performance from software written in a language that made no concessions to the vagaries of computer architecture led to work on compiler optimization in fruitful collaboration with researchers from IBM.
2. After studying Heinrich Behmann's algorithm for the decision problem of second order monadic predicate calculus, Jack saw that these methods could be extended to yield algorithms for decidable fragments of set theory. Over a period of decades, working with a group of collaborators almost all from Italy, who were first students at New York University and then became distinguished scientists in their own right, a surprising collection of non-trivial mathematics was found to lie within the scope of such algorithms.

3. Working with some of these same Italian researchers, a computer program was designed and implemented (at least as a prototype) that could verify the correctness of mathematical proofs presented in the language of set theory. Jack proposed to use this verifier to certify the correctness of a substantial body of the fundamentals of mathematical analysis. This was to include proofs of the basic properties of the real and complex number systems defined in set-theoretic terms, the fundamental properties of limits, continuity and the differential and integral calculus, and was to culminate in a proof of the Cauchy Integral Theorem of complex analysis.

The present volume is concerned with this verifier, its use and its context. However this context is to be understood in an extremely broad sense. Some of the work on decidable fragments of set theory is presented in a context that includes other algorithms from various sources for branches of logic as well. The main metamathematical theorems covered in a modern course in mathematical logic are here: the completeness theorem and the two incompleteness theorems of Gödel. Such topics as reflection principles and large cardinals are here as well. Those familiar with Jack Schwartz's mode of thought and with his way of putting his own stamp on a field will have no trouble hearing his voice in this important thought-provoking book.

Martin Davis

Professor Emeritus, Courant Institute, New York University
Visiting Scholar, University of California, Berkeley

Preface

In June 2000, the third named author visited New York University and was invited by Jack Schwartz to read what he called the ("common shared") *scenario*: a wide, carefully assembled sequence of definitions, theorems, and proofs, leading from the bare rudiments of set theory to the beginning of mathematical analysis. Proofs began to be gappy after a few hundred pages, and then totally absent, but the flow of definitions and theorems went on, to culminate in the definition of complex line integral and finally in the celebrated Cauchy integral theorem of complex analysis.

With an implementation appearing all but imminent, Jack had cast a significant piece of mathematics in rigorous formal detail, honestly asking himself whether a computer program could conceivably process and validate every single step. The resulting large-scale proof scenario was meant—in Jack's own words—"to serve as an essential part of the feasibility study that must precede the development of any ambitious proof-checker". Eventually, it would also serve as a testing-bench for the concrete implementation of the proof-checker.

The conception of this book on computational logic began then. According to our initial plans, the book would have described the structure of a proof verifier rooted in set theory and would also have surveyed a twenty-year long stream of results on decidable fragments of set theory.

One year later, the second named author visited New York in his turn; at his request Jack advanced the implementation work, speedily bringing into existence the proof-checker Referee, also known as Ref, or as ÆtnaNova. This is still a prototype, but it is reliable and fast enough to give us the possibility of debugging our proof scenarios. Some, though not all, of the content of this book is thus related to concrete experience, which we are now pleased to share with our readers.

A very large proof scenario is available today as a LaTeX-generated PDF-file. But given its size (over a thousand pages), it seems appropriate to publish it on the web (and eventually as a CD) rather than to print it. As for the proof verifier, it is usable on the web, but it depends on a SETL2 implementation. Since there is hardly anyone maintaining the SETL system today, Jack undertook with us a re-implementation of the proof verifier in a currently more popular language. But this will take some time; it should not be permitted to delay the publication of this book.

This is a posthumous publication, as its principal author passed away on March 2, 2009. In spite of his long illness, until the end of his life, Jack showed unbelievable resources of energy in the preparation of this book, in implementing and debugging Ref, and in drafting and writing the scenario. With the inspirer of this work gone, the book may not have achieved the degree of perfection that had been Jack Schwartz's goal; nevertheless we believe that the material he left behind will attract many readers and that its publication will be an appropriate tribute to a distinguished scientist.

A Word on the Audience for Whom This Book Is Intended

Any technical book must, by emphasizing certain details and leaving others unspoken, make definite assumptions about the prior knowledge of the reader.

This book assumes that the reader has a good knowledge of standard programming techniques, particularly of string manipulation and parsing, and also a general familiarity with those parts of mathematics that are analyzed in detail in the main series of definitions and proof scenarios to which much of the book is devoted.

On the other hand, little knowledge of formal logic is assumed. For this reason we try to present what is needed from logic in a reasonably self-contained way, emphasizing concepts likely to be important in continuations of the work begun here, rather than technicalities. Foundational issues, for example consideration of the strength or necessity of axioms, or the precise relationship of our formalism to other weaker or stronger formalisms studied in the literature, are neglected.

Because we expect our readers to be programmers of some sophistication, syntactic details of the kind that often appear early in books on logic are underplayed, and we repeatedly assume that anything programmable with relative ease can be taken as routine, and that the properties of such programmable operations can be proved when necessary to some theoretical discussion. This reflects our feeling that understanding develops top-down, focusing on details only as these become necessary.

We believe that too much detail is more likely to impede than to promote understanding. Who reads, or would want to read, the Whitehead–Russell *Principia*, or could testify that its hundreds of formula-filled pages are without error? But since we ask this question, why do we include hundreds of formula-filled pages in this book, which would not exist without the pioneering work of Whitehead and Russell? The reason lies in the fact that our formal proof text is, to a large extent, computer-checked. Though relatively useless to the human reader unless their correctness can be verified mechanically, long lists of formulae become useful once such verification becomes possible.

Content of This Book

Chapter 1 gives rapid overviews of the authors' approach to automated proof verification and of the large-scale formalized proof scenario whose development has accompanied the writing of this book.

Chapter 2 prepares for an extensive account of our proof verifier ÆtnaNova, by surveying three traditional branches of logic: propositional calculus, first-order predicate calculus, set theory. Completeness proofs are provided for the first two of these deductive systems; the much-debated issue of the consistency of Zermelo–Fraenkel set theory and of some of its proposed extensions is highlighted.

Chapter 3 provides an extended survey of inference mechanisms. Some of these belong to the initial endowment of ÆtnaNova, others are candidates for inclusion in that endowment should our proof verifier be re-implemented or enhanced. In some cases efficiency considerations show that an inference mechanism cannot be applied at its fullest; notwithstanding we present it because of the mathematical insight it provides. Two classics of the automated deduction field, Robinson's resolution principle and the Knuth–Bendix equational method, are also surveyed in this chapter.

Chapter 4 describes our verifier and its underlying design in more detail. In Chapter 5 we expand a broad survey of main definitions and theorems, showing the salient steps of a formalized proof scenario leading toward the (as yet) unachieved goal of proving the Cauchy integral theorem.

In Chapter 6, for completeness sake and to enjoy the intellectual insight that these results provide, we derive several of the main classical results on undecidability and unsolvability; in particular, Chaitin's theorem and the two celebrated Gödel's incompleteness theorems.

To convey the character of a scenario verifiable by means of our ÆtnaNova system, we conclude with Chapter 7 showing formalized proofs of many facts about ordinals, of various properties of the transitive closure operation, of finite and transfinite induction principles, and of Zorn's lemma.

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We are grateful to Martin Davis, to Alfredo Ferro, and to Alberto Policriti for encouraging, through decades of scientific interaction with the authors, the maturation of many ideas in this book; to Salvatore Paxia, who enabled us to keep the ÆtnaNova system alive; to Alexandru Ioan Tomescu, who contributed to the development of proof scenarios. Various results reported in this book stem from joint work with Gianluca Cincotti, Piero Ursino, and Calogero Zarba; Emanuele Giaquinta gave us precious advice concerning LaTeX.

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Catania, Italy — Domenico Cantone
Trieste, Italy — Eugenio G. Omodeo

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