Examples and Theorems in Analysis

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With 19 Figures

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Preface

Recent years have seen a number of worthy attempts to increase the accessibility of several branches of mathematics. In the library at AUS we find *A Friendly Introduction to Number Theory*, *An Adventurer's Guide to Number Theory*, and even *Rings and Things* (a volume of abstract algebra). In this vein we can anticipate *Rambling through Fields* (a sequel to the above), *Topology for Toddlers*, or perhaps *Euclidean Geometry on less than 200 Drachmas a Day.*

This said, it is not our intention now to write either *Trouble with Epsilons*, or *The Weekend Gardener's Guide to Uniform Convergence.* Mathematical analysis has its own intrinsic character, associated with inequalities and estimation, and with the balancing of one or more (potentially) small quantities – the epsilons of the above non-existent title. At the same time a number of the results concerning the processes of calculus – the chain rule for differentiation, for instance – are already familiar from previous informal courses and the instructor has to face the objection that 'we know all this; why are you repeating it in a more difficult way?'

To face these pedagogical difficulties is by no means easy, and we make no claim to have discovered the unique and infallible solution. The aim in this book is to try to give the subject concreteness and immediacy by giving the examples equal status with the theorems, as the title implies. The results are introduced and motivated by reference to examples which illustrate their use, as well as further examples to show how far the assumptions may be relaxed before the result fails. Indeed it is many of these examples which first arrest our attention, and perhaps even admiration. For a novice the sum of the geometric series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\left(\frac{-1}{2}\right)^n+\cdots=\frac{2}{3}$$

is interesting and a little unexpected, while the numerous classical series for π

such as Gregory's

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)}{2n+1}^{n}+\cdots=\frac{\pi}{4}$$

and Euler's

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$$

retain their surprise centuries after their discovery. At a more advanced level we can mention for instance the limit of the iterated sine

$$\lim_{n\to\infty}\sqrt{n}\sin_n(x)=\sqrt{3},\quad 0<x<\pi$$

(where $\sin_n(x)$ denotes the n-fold composition $\sin(\sin(\ldots\sin(x)))$) which is proved in Section 7.4, and the value of the theta function

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi}=\frac{\Gamma(1/4)}{\sqrt{2}\pi^{3/4}}$$

(which is discussed in Section 5.3 on the Gamma function) as formulae which give analysis its special interest, in addition to its numerous and powerful theoretical results.

The material of Chapters 1 to 6 is largely traditional, though with some novelties of presentation. We have chosen to base the proofs of results which are normally thought of as 'compactness arguments' on either the method of bisection or the existence of convergent subsequences. Indeed compactness – the existence of certain finite open coverings – belongs more in a course on metric spaces or topology rather than in elementary analysis on the real line.

We begin with sequences since there is only one small quantity to deal with and the structure of the definitions is clearer.

The discussion of Newton's method in Section 3.5 is a little more extended than usual, partly due to the author's exasperation with the common misunderstanding that if two terms in an iterative scheme agree to some number of decimal places, then they must equal the value of the limit to that accuracy.

We choose to treat integration using the set of regulated functions (those which have finite left- and right-hand limits at every point) since the technicalities are less than for the general Riemann approach, and at the same time the class of integrable functions is easily identified and sufficient for most uses at this level. For advanced applications, Lebesgue's theory and its generalisations are essential, but that is not our business here.

We consider improper integrals in some detail in Chapter 5 since many of the most interesting examples, for instance

$$\int_0^{\pi/2}\ln(\sin x)\,dx=-\frac{\pi}{2}\ln 2,\quad \int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$$

are of this type, and improper integrals motivate the discussion of distributions in Chapter 7.

We close with a number of applications in Chapter 7 since without them the question of the purpose of analysis goes largely unanswered. We have chosen Fourier theory, distributions and asymptotics, since they provide ideal settings in which to learn to use the results we have developed. Each of these topics has been the subject of long and detailed investigation and it is not our intention to be at all complete in our treatment; we shall simply attempt to indicate enough for the reader to see both what the subject is about, and what can be done with it.

The book *Introductory Mathematics: Algebra and Analysis* by Geoff Smith [14] is a useful preliminary to this one, and spares us the need for the customary 'Chapter 0'; in particular we shall refer to it for properties of the real and complex number systems.

The end of a proof is signalled as usual by ■; the end of the discussion following an example is signalled by ♦.

The exercises at the ends of chapters are at all levels with no particular indication of difficulty. Those marked with one star develop new ideas in some way, for instance convexity or the use of one-sided derivatives in relation to the Mean Value Theorem. Those with two stars are open questions – if you have solved one of these, please let the author know!

A book is rarely the product of a single person's efforts, and I should begin by thanking Professor John Toland, to whom I owe the original suggestion that I should write something in analysis for Springer UK. Professor Toland also showed me the striking proof of Chebyshev's theorem which is Exercise 13, Chapter 2. Two Dutch friends and colleagues, Dr. Adri Olde Daalhuis in Edinburgh and Dr. Jaap Geluk in Rotterdam read innumerable drafts with patience and good sense. They made valuable suggestions for additional results and examples, as well as helping me to avoid embarrassing errors. And most importantly the editorial staff of Springer UK were invariably helpful and patient with the long delays while I was involved with other projects. The text was prepared using Scientific Word.[1] The figures for which no other attribution is given were generated using Maple™.

To the student reader (who, it is said, never reads prefaces anyway) we shall say only that analysis is a challenge which will be rewarding in proportion to the amount of careful thinking which is devoted to it – Good Luck and Happy Problem Solving!

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