Graduate Texts in Mathematics 185

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Using Algebraic Geometry

Second Edition

With 24 Illustrations

Springer

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Preface to the Second Edition

Since the first edition of *Using Algebraic Geometry* was published in 1998, the field of computational algebraic geometry and its applications has developed rapidly. Many new results concerning topics we discussed have appeared. Moreover, a number of new introductory texts have been published. Our goals in this revision have been to update the references to reflect these additions to the literature, to add discussions of some new material, to improve some of the proofs, and to fix typographical errors. The major changes in this edition are the following:

- A unified discussion of how matrices can be used to specify monomial orders in §2 of Chapter 1.
- A rewritten presentation of the Mora normal form algorithm in §3 of Chapter 4 and the division of §4 into two sections.
- The addition of two sections in Chapter 8: §4 introduces the Gröbner fan of an ideal and §5 discusses the Gröbner Walk basis conversion algorithm.
- The replacement of §5 of Chapter 9 by a new Chapter 10 on the theory of order domains, associated codes, and the Berlekamp-Massey-Sakata decoding algorithm. The one-point geometric Goppa codes studied in the first edition are special cases of this construction.
- The Maple code has been updated and *Macaulay* has been replaced by *Macaulay 2*.

We would like to thank the many readers who helped us find typographical errors in the first edition. Special thanks go to Rainer Steinwandt for his heroic efforts. We also want to give particular thanks to Rex Agacy, Alicia Dickenstein, Dan Grayson, Serkan Hoşten, Christoph Kögl, Nick Loehr, Jim Madden, Mike O'Sullivan, Lyle Ramshaw, Hal Schenck, Hans Sterk, Mike Stillman, Bernd Sturmfels, and Irena Swanson for their help.

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Preface to the First Edition

In recent years, the discovery of new algorithms for dealing with polynomial equations, coupled with their implementation on inexpensive yet fast computers, has sparked a minor revolution in the study and practice of algebraic geometry. These algorithmic methods and techniques have also given rise to some exciting new applications of algebraic geometry.

One of the goals of *Using Algebraic Geometry* is to illustrate the many uses of algebraic geometry and to highlight the more recent applications of Gröbner bases and resultants. In order to do this, we also provide an introduction to some algebraic objects and techniques more advanced than one typically encounters in a first course, but which are nonetheless of great utility. Finally, we wanted to write a book which would be accessible to nonspecialists and to readers with a diverse range of backgrounds.

To keep the book reasonably short, we often have to refer to basic results in algebraic geometry without proof, although complete references are given. For readers learning algebraic geometry and Gröbner bases for the first time, we would recommend that they read this book in conjunction with one of the following introductions to these subjects:

- *Introduction to Gröbner Bases*, by Adams and Loustaunau [AL]
- *Gröbner Bases*, by Becker and Weispfenning [BW]
- *Ideals, Varieties and Algorithms*, by Cox, Little and O'Shea [CLO]

We have tried, on the other hand, to keep the exposition self-contained outside of references to these introductory texts. We have made no effort at completeness, and have not hesitated to point the reader to the research literature for more information.

Later in the preface we will give a brief summary of what our book covers.

The Level of the Text

This book is written at the graduate level and hence assumes the reader knows the material covered in standard undergraduate courses, including abstract algebra.

But because the text is intended for beginning graduate students, it does not require graduate algebra, and in particular, the book does not assume that the reader is familiar with modules. Being a graduate text, *Using Algebraic Geometry* covers more sophisticated topics and has a denser exposition than most undergraduate texts, including our previous book [CLO].

However, it is possible to use this book at the undergraduate level, provided proper precautions are taken. With the exception of the first two chapters, we found that most undergraduates needed help reading preliminary versions of the text. That said, if one supplements the other chapters with simpler exercises and fuller explanations, many of the applications we cover make good topics for an upper-level undergraduate applied algebra course. Similarly, the book could also be used for reading courses or senior theses at this level. We hope that our book will encourage instructors to find creative ways for involving advanced undergraduates in this wonderful mathematics.

How to Use the Text

The book covers a variety of topics, which can be grouped roughly as follows:

- Chapters 1 and 2: Gröbner bases, including basic definitions, algorithms and theorems, together with solving equations, eigenvalue methods, and solutions over $\mathbb{R}$.
- Chapters 3 and 7: Resultants, including multipolynomial and sparse resultants as well as their relation to polytopes, mixed volumes, toric varieties, and solving equations.
- Chapters 4, 5 and 6: Commutative algebra, including local rings, standard bases, modules, syzygies, free resolutions, Hilbert functions and geometric applications.
- Chapters 8 and 9: Applications, including integer programming, combinatorics, polynomial splines, and algebraic coding theory.

One unusual feature of the book's organization is the early introduction of resultants in Chapter 3. This is because there are many applications where resultant methods are much more efficient than Gröbner basis methods. While Gröbner basis methods have had a greater theoretical impact on algebraic geometry, resultants appear to have an advantage when it comes to practical applications. There is also some lovely mathematics connected with resultants.

There is a large degree of independence among most chapters of the book. This implies that there are many ways the book can be used in teaching a course. Since there is more material than can be covered in one semester, some choices are necessary. Here are three examples of how to structure a course using our text.

- Solving Equations. This course would focus on the use of Gröbner bases and resultants to solve systems of polynomial equations. Chapters 1, 2, 3

and 7 would form the heart of the course. Special emphasis would be placed on §5 of Chapter 2, §5 and §6 of Chapter 3, and §6 of Chapter 7. Optional topics would include §1 and §2 of Chapter 4, which discuss multiplicities.

- Commutative Algebra. Here, the focus would be on topics from classical commutative algebra. The course would follow Chapters 1, 2, 4, 5 and 6, skipping only those parts of §2 of Chapter 4 which deal with resultants. The final section of Chapter 6 is a nice ending point for the course.

- Applications. A course concentrating on applications would cover integer programming, combinatorics, splines and coding theory. After a quick trip through Chapters 1 and 2, the main focus would be Chapters 8 and 9. Chapter 8 uses some ideas about polytopes from §1 of Chapter 7, and modules appear naturally in Chapters 8 and 9. Hence the first two sections of Chapter 5 would need to be covered. Also, Chapters 8 and 9 use Hilbert functions, which can be found in either Chapter 6 of this book or Chapter 9 of [CLO].

We want to emphasize that these are only three of many ways of using the text. We would be very interested in hearing from instructors who have found other paths through the book.

References

References to the bibliography at the end of the book are by the first three letters of the author's last name (e.g., [Hil] for Hilbert), with numbers for multiple papers by the same author (e.g., [Mac1] for the first paper by Macaulay). When there is more than one author, the first letters of the authors' last names are used (e.g., [AM] for Atiyah and Macdonald), and when several sets of authors have the same initials, other letters are used to distinguish them (e.g., [BoF] is by Bonnesen and Fenchel, while [BuF] is by Burden and Faires).

The bibliography lists books alphabetically by the full author's name, followed (if applicable) by any coauthors. This means, for instance, that [BS] by Billera and Sturmfels is listed before [Bla] by Blahut.

Comments and Corrections

We encourage comments, criticism, and corrections. Please send them to any of us:

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For each new typo or error, we will pay $1 to the first person who reports it to us. We also encourage readers to check out the web site for *Using Algebraic Geometry*, which is at

`http://www.cs.amherst.edu/~dac/uag.html`

This site includes updates and errata sheets, as well as links to other sites of interest.

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