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Automorphisms of Finite Groups

Springer

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ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-981-13-2894-7 ISBN 978-981-13-2895-4 (eBook)
https://doi.org/10.1007/978-981-13-2895-4

Library of Congress Control Number: 2018957651

AMS Subject Classification (2010): 20D45, 20J05, 20D15, 20E18

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The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

In mathematics, the theory of groups is a basic tool for the study of symmetries of objects. From this point of view, the symmetries of a group G itself are encoded in the automorphism group Aut(G) of G. Thus, the automorphism groups of finite groups are of fundamental interest in the study of the theory of groups. Since finite groups are in abundance, a complete exposition of all aspects of automorphism groups of finite groups is not feasible in a single volume. Thus, exposition of developments according to a specific theme is highly justifiable. The purpose of this monograph is to present developments on the relationship between orders of finite groups and those of their automorphism groups.

It is clear that the automorphism group of a finite group is finite. Therefore, it is natural to look for a relationship between the order $|G|$ of the group G and the order $|\mathrm{Aut}(G)|$ of its automorphism group Aut(G). Over the years, this has been studied extensively. The first-known result regarding this problem is due to Frobenius [36], an exposition of which is also available in [16, pp. 250–252], where it is shown that the order of G controls the orders of the elements in Aut(G). Birkhoff and Hall [12] proved that $|\mathrm{Aut}(G)|$ is always a divisor of $|\mathrm{Aut}(E)||G|^{r-1}$, where r is the number of distinct prime divisors of $|G|$ and E is the elementary abelian group of order $|G|$. In the direction of a lower bound on $|\mathrm{Aut}(G)|$, Hilton [58] proved that if G is a finite abelian group such that p^n divides $|G|$, p prime, then $p^{n-1}(p-1)$ divides $|\mathrm{Aut}(G)|$. Continuing this line of investigation, Herstein and Adney [55] proved that if G is any finite group such that p^2 divides $|G|$, then p divides $|\mathrm{Aut}(G)|$. In 1954, Scott [114] proved that if p^3 divides $|G|$, then p^2 divides $|\mathrm{Aut}(G)|$. With these results as motivation, Scott [114] conjectured that a finite group has at least a prescribed number of automorphisms if the order of the group is sufficiently large. The local version of the conjecture is as follows:

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for each $h \in \mathbb{N}$ and each prime p, if G is any finite group such that $p^{f(h)}$ divides $|G|$, then p^h divides $|\mathrm{Aut}(G)|$.

In 1956, Ledermann and Neumann [80, 81] confirmed the above conjecture by constructing a cubic polynomial function with the desired properties, using extension theory of groups and bounds on the order of Schur multiplier of finite groups. We, henceforth, refer to the above statement as Ledermann–Neumann theorem. Green [49] improved previously known bounds on the order of Schur multiplier of finite p-groups and, using it together with the standard factorization of 2-cocycle associated to a central extension of groups, refined the functional bound of Ledermann–Neumann theorem to a quadratic polynomial function. Howarth [63] and Hyde [68] further improved this quadratic polynomial function by deriving new bounds on the order of the group of central automorphisms. One of the aims of the monograph is to give an exposition of these works. We have chosen to present the various improvements rather than just the best-known results in view of the new group-theoretic results of independent interest obtained in the course of improving the bound.

It is natural to ask whether the function f can be bounded from below. Hyde [68] provided an answer to this question by showing that the least function f such that $|\mathrm{Aut}(G)|_p \geq p^h$ whenever $|G|_p \geq p^{f(h)}$ satisfies $f(h) \geq 2h - 1$. One might wonder whether this bound can be further lowered down if we restrict ourselves to the class of finite p-groups.

For every group G, the subset $\mathrm{Inn}(G)$ consisting of all inner automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$. Thus, we can construct the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$, denoted by $\mathrm{Out}(G)$, and called the group of outer automorphisms of G. It seems that Schenkman [108] was the first to ask the following question:

Does every non-abelian finite p-group admit a non-inner automorphism of prime power order?

In the same paper, he claimed a much stronger statement for groups of nilpotency class 2: Namely, if G is a finite p-group of nilpotency class 2, then $|G|$ divides $|\mathrm{Aut}(G)|$. Unfortunately, there remained a gap in his proof. A correct proof of the preceding statement was later given by Faudree [33]. The question was answered in full generality by Gaschütz [40] in 1966. Using cohomological methods, he proved that if G is a finite p-group of order at least p^2, then p divides $|\mathrm{Out}(G)|$. Thus, if a non-abelian group G is of order p^n and its center $\mathrm{Z}(G)$ is of order p, then by the preceding result of Gaschütz, $|G|$ divides $|\mathrm{Aut}(G)|$. In the same year, Otto [98], motivated by the functional bounds due to Herstein and Adney [55] and Scott [114], proved that if G is a non-cyclic finite abelian p-group of order greater than p^2, then $|G|$ divides $|\mathrm{Aut}(G)|$. By this time, the following problem, which was later also recorded in [89, Problem 12.77], became well known.

Divisibility Problem: *Does the order of a non-cyclic finite p-group G of order greater than p^2 divide the order of its automorphism group* $\mathrm{Aut}(G)$*?*

Notice that Divisibility Problem can also be thought of as a question whether for all non-cyclic finite p-groups G, the identity function f on $\mathbb{N}\backslash\{1,2\}$ satisfies the property that, for each $h \in \mathbb{N}\backslash\{1,2\}$, if $p^{f(h)}$ divides $|G|$, then p^h divides $|\mathrm{Aut}(G)|$.

One of the first reductions for Divisibility Problem was given by Otto [98], who showed that it is enough to consider purely non-abelian p-groups (recall that a group is said to be *purely non-abelian* if it does not admit a nontrivial abelian direct factor). Buckley [14] reduced the problem further by showing that we can restrict our attention to only those finite p-groups for which the center $\mathrm{Z}(G)$ is contained in the Frattini subgroup $\Phi(G)$ of G. It has since been proved that the problem has an affirmative solution for many special classes of non-cyclic finite p-groups, for example p-groups with metacyclic central quotient [18], modular p-groups [22], p-abelian p-groups [19], groups with small central quotient [20], groups with cyclic Frattini subgroup [30], groups of order p^7 [41], and groups of coclass 2 [35]. Using techniques from coclass theory, Eick [26] showed that, for all but finitely many 2-groups of a given coclass, the problem has an affirmative solution.

It is worth pointing out that the main ingredient in the solution of Divisibility Problem for groups G in various classes is the subgroup

$$\mathrm{IC}(G) := \mathrm{Inn}(G)\mathrm{Autcent}(G)$$

of the automorphism group $\mathrm{Aut}(G)$ of G, where

$$\mathrm{Autcent}(G) = \{\phi \in \mathrm{Aut}(G) \mid x^{-1}\phi(x) \in \mathrm{Z}(G) \text{ for all } x \in G\},$$

the group of central automorphisms of G.

In 2015, it was shown by González-Sánchez and Jaikin-Zapirain [46], making an extensive use of pro-p-techniques, that there exist non-cyclic finite p-groups of order greater than p^2 for which $|G|$ does not divide $|\mathrm{Aut}(G)|$. We present a detailed exposition of this important development in the theory of automorphism groups. However, it is still intriguing to know for what other classes of non-cyclic finite p-groups the problem has an affirmative solution and to construct explicit counterexamples.

In view of the existence of counterexamples to Divisibility Problem, it is reasonable to introduce the following property:

A non-cyclic finite p-group G of order greater than p^2 *is said to have Divisibility Property if* $|G|$ *divides* $|\mathrm{Aut}(G)|$.

Determining all finite p-groups admitting Divisibility Property continues to be a challenging problem.

Another fundamental problem in the study of automorphism groups of finite groups is the following extension and lifting problem for automorphisms of groups.

If $\mathcal{E} : 1 \to N \to G \to H \to 1$ *is a short exact sequence of groups, then under what conditions does an automorphism of N extend to an automorphism of G and analogously when does an automorphism of H lift to an automorphism of G?*

A crucial role in the investigation of this problem has been played by an exact sequence due to Wells [126] relating derivations, automorphisms, and second cohomology groups of groups under consideration. Let

$$\alpha : H \to \mathrm{Out}(N)$$

be the coupling associated to the extension $\mathcal{E}$ (see Sect. 2.2 of Chap. 2 for definition). The coupling α leads to an H-module structure on the center $\mathrm{Z}(N)$ of N. Let $\mathrm{Der}(H, \mathrm{Z}(N))$ be the group of derivations from H to $\mathrm{Z}(N)$, $\mathrm{Aut}_N(G)$ the group of automorphisms of G normalizing N (keeping N invariant as a set),

$$C_\alpha = \{(\phi, \theta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(N) \mid \bar{\theta}\,\alpha(x)\,\bar{\theta}^{-1} = \alpha(\phi(x)) \text{ for all } x \in H\},$$

and $\mathrm{H}^2(H,\ \mathrm{Z}(N))$ the second cohomology of H with coefficients in $\mathrm{Z}(N)$, where $\bar{\theta} = \mathrm{Inn}(N)\theta \in \mathrm{Out}(N)$. With the foregoing setting, Wells derived the following fundamental exact sequence, which we refer to as Wells exact sequence throughout the monograph:

$$1 \to \mathrm{Der}(H,\ \mathrm{Z}(N)) \to \mathrm{Aut}_N(G) \to C_\alpha \to \mathrm{H}^2(H,\ \mathrm{Z}(N)).$$

Wells exact sequence also turned out to be very useful, as we will see, in the study of the relationship between the order of a finite group and the order of its automorphism group. Notably, Wells exact sequence is used in Sect. 4.1 of Chap. 4 to obtain a reduction of Divisibility Problem for finite p-groups to the case when $\mathrm{Z}(G) \leq \Phi(G)$. Another instance of an extensive use of Wells exact sequence is in Sect. 5.3 of Chap. 5 for estimating $|\mathrm{Aut}_N(G)|$ for 2-groups G. Notice that the exactness of Wells sequence at C_α is equivalent to saying that a pair of automorphisms in C_α is induced by an automorphism in $\mathrm{Aut}_N(G)$ if and only if the cohomology class corresponding to the pair vanishes. This fact is used in many instances throughout the monograph to extend certain automorphism of N to an automorphism of G, for instance, Sects. 3.3 and 3.4 in Chap. 3 and Sect. 5.4 in Chap. 5.

Expositions of Wells exact sequence have appeared in the literature a couple of times, for example, by Robinson [104, 105] and recently by Dietz [23]. The extension and lifting problem for automorphisms has been investigated using Wells exact sequence by Jin [73], Passi, Singh and Yadav [99], and Robinson [106, 107]. In the monograph, we give a thorough exposition of Wells exact sequence following its construction due to Jin-Liu [74], along with some applications.

The monograph is broadly divided into three parts. The first part is an exposition of Wells exact sequence including some of its applications. The second part is an exposition of various developments on the functional bound. The final part presents the works on Divisibility Property of finite p-groups culminating in the existence of groups without this property.

In Chap. 1, we discuss the basic results on p-groups that are used in subsequent chapters of the monograph.

In Chap. 2, we begin with basic notions from group cohomology and extension theory of groups leading to the construction of the fundamental exact sequence of Wells. We also discuss some ramifications of Wells exact sequence, followed by a characterization of extensions with trivial coupling in terms of central products of groups. We conclude the chapter with applications of Wells exact sequence in the extension and lifting problem for automorphisms of finite groups.

In Chap. 3, starting with basic results on Schur multiplier of finite groups, we present the Ledermann–Neumann theorem giving a cubic polynomial function and its subsequent refinements to quadratic polynomial functions. In particular, an affirmative solution of Divisibility Problem for non-cyclic abelian p-groups is derived.

In Chap. 4, we take up the study of groups admitting Divisibility Property and first present some general reduction results allowing us to work in a smaller class of finite p-groups, namely the class of purely non-abelian finite p-groups. We then discuss Divisibility Property for (i) p-groups of nilpotency class 2, (ii) p-groups with metacyclic central quotient, (iii) modular p-groups, (iv) p-abelian p-groups, and (v) p-groups with small central quotient.

In Chap. 5, we continue our study of Divisibility Property and examine it for (i) p-groups of order p^7, (ii) p-groups of coclass 2, (iii) p-groups of a given coclass, and (iv) p^2-abelian p-central p-groups. Finally, we present results for some other classes of groups under some rather strong conditions, including p-groups with cyclic Frattini subgroup.

In Chap. 6, which is of slightly different flavor, we present the existence of groups without Divisibility Property using Lie theory and pro-p-techniques.

The monograph is aimed at researchers working in group theory; particularly, graduate students in algebra will hopefully find it useful. The primary prerequisite is an advanced course in the theory of finite groups. Section 5.3 of Chap. 5 requires familiarity with coclass theory and pro-p-groups. Additionally, Chap. 6 demands familiarity with basic analysis, topology and Lie algebras. An attempt has been made to unify various ideas developed over the years and to keep the monograph mostly self-contained. Either the results are proved or complete references are provided. Also, some problems are posed at appropriate places, more precisely Problems 2.57, 3.50, 3.88, 4.22, 4.35, 5.11, and 6.22.

We conclude with some notational setup. Throughout the monograph, we evaluate the composition of functions from right to left. For example, if $f : A \to B$ and $h : B \to C$ are two functions, then $hf(x) = h(f(x))$. The internal direct product of two subgroups H and K of a multiplicatively written group G (not necessarily abelian) is denoted by $H \oplus K$.

Acknowledgements

Inder Bir Singh Passi thanks Indian National Science Academy, New Delhi, for their support and Ashoka University, Sonipat, for making available their facilities. Mahender Singh acknowledges support from the DST INSPIRE Faculty grant and the SERB MATRICS grant. Manoj Kumar Yadav thanks the Indian Institute of Science Education and Research, Mohali for providing excellent hospitality during his visits to the Institute. Inder Bir Singh Passi and Mahender Singh thank Harish-Chandra Research Institute, Allahabad, for warm hospitality extended by the Institute during their visits. Mahender Singh and Manoj Kumar Yadav thank Dipendra Prasad for hosting them at the Tata Institute of Fundamental Research, Mumbai, for a visit while this work was in progress. The authors thank Charles Leedham-Green and Rahul D. Kitture for their useful comments. The authors are grateful to Shamim Ahmad and Krati Shrivastav from Springer for their untiring support at the production stage of the monograph.

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Notation

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Ring of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{C}^{\times}$	Multiplicative group of nonzero complex numbers
$\mathbb{F}_q$	Finite field with q-elements
$\mathbb{Z}_p$	Ring of p-adic integers
$\mathbb{Q}_p$	Field of p-adic numbers
$\lfloor x \rfloor$	Greatest integer less than or equal to the number x
$\lceil x \rceil$	Smallest integer greater than or equal to the number x
n_p	Highest power of p that divides n
(a, b)	Greatest common divisor of integers a and b
φ	Euler's totient function
$f\vert_A$	Restriction of the map f on a subset A of its domain
C_n	Cyclic group of order n
$\mathbb{Z}/n\mathbb{Z}$	Cyclic group of integers modulo n
Σ_n	Symmetric group on n symbols
$\exp(G)$	Exponent of G
$d(G)$	Minimum number of generators of G
$\langle X \rangle$	Subgroup of a group G generated by a subset X
HK	$\{hk \mid h \in H, k \in k\}$, where H and K are subgroups of a group G
$\mathrm{cc}(G)$	Coclass of G
$\vert G \vert$	Order of G
$\vert g \vert$	Order of $g \in G$ or equivalently $\vert\langle g \rangle\vert$
$\mathrm{Ker}(\phi)$	Kernel of the group homomorphism $\phi : G \to H$
$\mathrm{Im}(\phi)$	Image of the group homomorphism $\phi : G \to H$
${}^{\phi}x$	Image of $x \in X$ under the map $\phi : X \to Y$
${}^{G}x$	Conjugacy class of x in G
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G

$\|G : H\|$	Index of the subgroup H in G
$N \trianglelefteq G$	N is a normal subgroup of G
$N \triangleleft G$	N is a proper normal subgroup of G
$X \backslash Y$	Set of elements of X which are not in Y
$[x_1, x_2]$	Commutator $x_1 x_2 x_1^{-1} x_2^{-1}$
$[x_1, \ldots, x_n]$	$[\ldots[[x_1, x_2], x_3], \ldots, x_n]$ the left-normed commutator of $x_1, \ldots, x_n$
$[H, K]$	Subgroup of G generated by $\{[h, k] \mid h \in H \, and \, k \in K\}$, where H and K are subgroups of G
$[G, G]$	Commutator subgroup of G
$\mathrm{Z}(G)$	Center of G
$\mathrm{C}_G(H)$	Centralizer of H in G
$\mathrm{C}_G(x)$	Centralizer of $\langle x \rangle$ in G
$\mathrm{N}_G(H)$	Normalizer of H in G
$\gamma_n(G)$	nth term of the lower central series of G; in particular, $\gamma_2(G) = [G, G]$
$\mathrm{Z}_n(G)$	nth term of the upper central series of G; in particular, $\mathrm{Z}_1(G) = \mathrm{Z}(G)$
$\Omega_n(G)$	$\langle x \in G \mid x^{p^n} = 1 \rangle$
$\mho_n(G)$	$\langle x^{p^n} \mid x \in G \rangle$
A_p	Unique Sylow p-subgroup of the abelian group A
$\Phi(G)$	Frattini subgroup of G
$\mathrm{Aut}(G)$	Group of automorphisms of G
$\mathrm{Inn}(G)$	Group of inner automorphisms of G
$\mathrm{Out}(G)$	$\mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\mathrm{Aut}^H(G)$	Group of automorphisms of G which centralize the subgroup H (i.e., act as identity automorphism on H) or in case H is a quotient group of G, induce identity on H
$\mathrm{Aut}_H(G)$	Group of automorphisms of G which normalize the subgroup H (i.e., keep H invariant as a set)
$\mathrm{Aut}_K^H(G)$	$\mathrm{Aut}_K(G) \cap \mathrm{Aut}^H(G)$
$\mathrm{Aut}^{K,H}(G)$	$\mathrm{Aut}^K(G) \cap \mathrm{Aut}^H(G)$
$\mathrm{Autcent}(G)$	$\{\phi \in \mathrm{Aut}(G) \| x^{-1}\phi(x) \in \mathrm{Z}(G) \, \text{for all} \, x \in G\}$ or equivalently $\mathrm{Aut}^{G/\mathrm{Z}(G)}(G)$
$\mathrm{C}_{\mathrm{Out}(G)}(\mathrm{Z}(G))$	$\mathrm{Aut}^{\mathrm{Z}(G)}(G)/\mathrm{Inn}(G)$
$\mathrm{IC}(G)$	$\mathrm{Inn}(G)\mathrm{Autcent}(G)$
ι_x	Inner automorphism of G induced by $x \in G$
X^G	$\{x \in X \mid {}^g x = x \, \text{for all} \, g \in G\}$, the fixed-point set of G-action on X
G_x	$\{g \in G \mid {}^g x = x\}$, the stabilizer subgroup of G at $x \in X$
$G \oplus H$	Direct product of G and H
$\oplus_i G_i$	Direct product of the family $\{G_i\}$
$X \times Y$	Cartesian product of X and Y
$\prod_i X_i$	Cartesian product of the family $\{X_i\}$
$\det(A)$	Determinant of a square matrix A

$\mathrm{M}(r, R)$	Ring of $r \times r$ matrices with entries in the ring R
$\mathrm{GL}(r, R)$	Group of $r \times r$ invertible matrices with entries in the ring R
$\mathrm{O}(r, R)$	Group of $r \times r$ orthogonal matrices with entries in the ring R
$\mathrm{UT}(r, R)$	Group of $r \times r$ upper unitriangular matrices with entries in the ring R
$\mathcal{M}(G)$	Schur multiplier of G
$R[[X]]$	Ring of formal power series in X with coefficients in the ring R
$R[G]$	Group algebra of G over the ring R
$\mathbb{Z}_p[[G]]$	Completed group algebra of a pro-p-group G over the ring $\mathbb{Z}_p$
$I_R(G)$	Augmentation ideal of the group algebra $R[G]$
$\mathrm{Der}(G, A)$	Group of derivations from G to A, where A is a G-module
$\mathrm{Der}(L)$	Derivation algebra of the Lie algebra L
$\mathrm{InnDer}(L)$	Inner derivation algebra of the Lie algebra L
$\mathrm{H}^n(G, A)$	nth cohomology of G with coefficients in the G-module A
$\mathrm{H}^n_{cts}(G, A)$	nth continuous cohomology of the topological group G with coefficients in the topological G-module A
$\mathrm{H}^n_{Lie}(L, A)$	nth Lie algebra cohomology of L with coefficients in the L-module A
$\mathrm{Ext}(H, N)$	Set of equivalence classes of extensions of H by N
$\mathrm{Ext}_\alpha(H, N)$	Set of equivalence classes of extensions of H by N inducing the coupling $\alpha : H \to \mathrm{Out}(N)$
$\Lambda^k L$	kth exterior power of the Lie algebra L
$\mathrm{Hom}(G, A)$	Group of homomorphisms from G to the abelian group A
$\mathrm{Hom}_R(M, N)$	Module of R-module homomorphisms from M to N
$\mathrm{End}_R(M)$	Algebra of R-module endomorphisms of M
$\widehat{G}$	Character group $\mathrm{Hom}(G, \mathbb{C}^\times)$ of G
$M \otimes_R N$	Tensor product of R-modules M and N