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Theory of Besov Spaces

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Dedicated to my family, Ema and Emiri.

Preface

This book is intended to be as an exhaustive and self-contained treatment of Besov spaces and Triebel–Lizorkin spaces. In this book I aim to explain Besov spaces and Triebel–Lizorkin spaces from the start and to apply them to partial differential equations (PDEs).

I mean here by "a function space" a linear subspace of the space of all functions in a set X. We will often work in the setting of $\mathbb{R}^n$ but in the last part of the book we consider open sets in $\mathbb{R}^n$. For example, the set of all functions on $X = \mathbb{R}^n$ is an example of function spaces. However, it is too vague and too hard to grasp. As more typical examples, let us consider the subspace of all Borel measurable functions, in particular the space of all continuous functions. Or, readers may envisage the space $\mathrm{BC}(\mathbb{R}^n)$ the set of all bounded continuous functions which readers will have encountered in lectures on topological spaces. If readers are familiar with the theory of integrals, we can think that we are going to study a new framework containing the set of all integrable or measurable functions.

There are too many continuous or measurable functions and, therefore, their various linear subspaces. In this book, we are going to propose new frameworks called Besov spaces and Triebel–Lizorkin spaces and then discuss their properties in detail.

When we were high school students or undergraduate students, our main concern lay in the C^∞-functions. Therefore, readers cannot understand why we are going to investigate functions in a very subtle manner. However, there do exist many examples where the non-differentiable functions play a fundamental role in the rule of the nature.

As an example, I take up Brownian motion. This is familiar because we learnt in chemistry lessons that this describes the motion of particles. Here by Brownian motion, I mean the "mathematical" one which grew out of the chemical one. This mathematical Brownian motion is a fundamental concept of stochastic integrals initiated by Kiyosi Itô and hence we are convinced that the mathematical Brownian motion plays a fundamental role in economics. As well as the chemical one, the mathematical one moves in a very complicated way. Namely, each path is continuous but it is too complicated and non-differentiable. When we want to describe properties of continuous functions which are not differentiable, Besov spaces and Triebel–Lizorkin spaces are useful. As this example shows, Besov spaces and Triebel–Lizorkin spaces play an important role.

Another aim of this book is to apply Besov spaces and Triebel–Lizorkin spaces to PDEs.

In PDEs, we are led to consider the equations beyond the framework of $C^k(\mathbb{R}^n)$ functions. As the example of the functions

$$f(x, y) = (x^2 - y^2) \log\log \frac{1}{x^2 + y^2} \in C^2(B(1) \setminus \{(0, 0)\}),$$

defined in $B(1)$, together with computation

$$\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = \frac{4(x^2 - y^2)(2\log(x^2 + y^2) - 1)}{(x^2 + y^2)(\log(x^2 + y^2))^2} \in C^2(B(1))$$

shows, it does not suffice to consider functions which are k-times differentiable and whose partial derivatives up to order k are all continuous when we consider the Poisson equation $-\Delta u = f$. For example, when we consider the elliptic differential equations, we use not C^2 but $\mathscr{C}^{2+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon \in (0, 1)$, where $\mathscr{C}^{2+\varepsilon}(\mathbb{R}^n)$ denotes the Hölder–Zygmund space of order $2 + \varepsilon$. As we establish later in this book, the space $\mathscr{C}^{2+\varepsilon}(\mathbb{R}^n)$ is realized as a special case of Besov spaces and Triebel–Lizorkin spaces. Hence we see that Besov spaces and Triebel–Lizorkin spaces are useful in PDEs. The branch of PDE being too wide, we cannot take up all of them, but we seek to investigate the wave equations, the Schrödinger equations, the heat equations, and the elliptic differential equations in the context of applications of Besov spaces and Triebel–Lizorkin spaces.

Let us consider the complexity and what we obtain from that. We learnt in high school that $f : \mathbb{R} \to \mathbb{R}$ is convex if f is twice differentiable, and $f'' \geq 0$. However, in undergraduate course, this is equivalent to $f((1-t)x+ty) \leq (1-t)f(x)+tf(y)$ for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$. With this in mind, let us reconsider the definition of convex functions and discuss the properties of this notion.

First of all, compare "$f'' \geq 0$" with "$f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$ for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$"; occasionally the latter is not easy to prove. We can clearly say that the former is simple. However, the latter enjoys properties that the former does not enjoy. For example, due to the fact that the differentiation comes into play and that the latter is readily extended to the functions on linear spaces, the former is harder to generalize than the latter. Thus, we can say that the latter is general. In addition, when we define the convexity by way of the differentiation, we cannot say that the convexity is stable under taking the modulus. Also, in entrance examinations, we frequently encounter inequalities of the form $f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$ because this inequality contains three parameters x, y, and t. Thus, once $f((1 - t)x + ty) \leq (1 - t)f(x) + tf(y)$ is proved, it should be useful.

For PDEs, we must be familiar with functional analysis. However, we need not be so serious: we provide some facts on functional analysis in this book and this is sufficient.

Many mathematicians hate Besov spaces and Triebel–Lizorkin spaces. Why? I now consider the reason here. As I mentioned earlier, there are many function spaces. Although I content myself with listing them briefly, there are $L^p(\mathbb{R}^n)$-spaces, Sobolev spaces, Morrey spaces, Orlicz spaces, Besov spaces, Triebel–Lizorkin spaces, and so on. Among them I am led to the following conclusion: The easier to define the function spaces are, the fewer good properties they have. Ideally, from a good notion of function spaces, we want that: it is simple enough to memorize the definition and it has a rich structure. The spaces $L^p(\mathbb{R}^n)$-spaces, Sobolev spaces, Morrey spaces, Orlicz spaces, which I listed earlier, are easy to describe but they cannot cover differentiability and integrability. Meanwhile, Besov spaces and Triebel–Lizorkin spaces describe very well differentiability and integrability but their definitions are very complicated. For example, we can consider Besov-Morrey spaces which are defined by mixing Besov spaces and Morrey spaces. You can easily guess that the definition of Besov-Morrey spaces is very complicated. Many people hate Besov spaces and Triebel–Lizorkin spaces because of the complexity of the definition which arises as a price of the good properties, I think.

However, let me stress that these function spaces have many big advantages. As I mentioned before, it is important to be able to grasp many other function spaces. For example, $L^p(\mathbb{R}^n)$-spaces, the BMO space, Sobolev spaces, and Hardy spaces fall under the unified framework of Besov spaces and Triebel–Lizorkin spaces. Here I content myself with mentioning that the atomic decomposition is extremely important. For we can learn much more once we establish the theory of the atomic decomposition. The details are left to Chap. 5.

Let me describe the structure of this book while comparing the content of this book with the existing literature. We have to prepare a lot in order to define Besov spaces and Triebel–Lizorkin spaces and in order to establish the theory of these spaces because the definitions of Besov spaces and Triebel–Lizorkin spaces are very complicated. So in Chap. 1, I explain the Fourier transform, the maximal operator, and the singular integral operators, which are of fundamental importance in harmonic analysis. The theory of singular integral operators being too wide with a large amount of literature, I content myself with its brief introduction. I also kept the description of the singular integral operators to the minimum. See [22, 31, 32, 86] for more about the theory of singular integral operators. In Chap. 2, for the purpose of a survey of the book, I investigate $B^s_{pq}(\mathbb{R}^n)$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. If we are limited to $B^s_{pq}(\mathbb{R}^n)$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, we definitely want to make it clear that we can define the space and investigate the property without using any heavy tool except $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$. Chapter 2 presents one of my primary aims in this book. We want to define Besov spaces and Triebel–Lizorkin spaces and then we investigate fundamental properties: density of the spaces, completeness of the spaces, and so on. As a by-product of the Plancherel–Polya–Nikolskii inequality, I described the modulation spaces. See [35] for the exhaustive description of the time frequency analysis. There are recent textbooks dealing with Besov spaces. For example Grafakos gave a description in [33]. If I compare at this book with my book, I believe that some equivalent expressions are described in more detail. Chapter 3 is devoted to showing that these spaces cover many other spaces as special cases. This chapter somewhat overlaps [22, 31, 32, 86] and supplements [97, 99]. We paid attention to dyadic analysis which has developed rapidly in this decade. Atomic decomposition is taken up in Chap. 4 together with applications to the boundedness of operators. Bourdaud has written a book on the paraproduct [12]. I did manage to formulate the model cases. For more about the paraproduct see [12]. Ma'zya dealt with Sobolev spaces in the context of multipliers [57, 58]. Applications to PDEs are contained in Chap. 5. Besov together with his collaborators has written books [8, 9]. Although Besov and I dealt commonly with the function spaces defined on domains, I can say that Besov concentrated more on the function spaces on domains, while I placed myself mainly in the Euclidean space. Recently theory of function spaces underwent a major diversification. So, in Chap. 6 we describe how what we have been doing can be generalized and applied. Each section of the last chapter will play a role of an introduction of various function spaces and it will also play the role of the brief introduction of the subject. See [1] for Morrey spaces, [16] for variable Lebesgue spaces, and [51] Weighted Sobolev spaces. Triebel has written many recent books [106–110]. The books [23, 101, 106] deal with fractals. I will content myself with introducing the function spaces. My book does not overlap [107, 108]. In [109, 110] Triebel dealt with Morrey-type function spaces. In this book, I mentioned briefly this type of function spaces in a different context. As an application, I chose the solution to the Kato problem. However, I could manage to allude to its solution and some related facts. See the book [5] for more exhaustive details.

The key theorems in this book are Theorem 1.49 and its corollary Theorem 1.53, as well as Theorem 2.34 and its corollary Theorem 4.1. Needless to say, the definition is the most important in mathematics. However, these four theorems appear repeatedly and we cannot follow the proof without them. By this I do not mean that we have to memorize these theorems but we will use them quite often.

Finally, we describe how different this book is from the existing literature. As is mentioned above, we heavily depend on Theorems 1.49, 1.53, 2.34, and 4.1. Among the textbooks I listed in the references the first two theorems are exhaustively investigated in [99] as well as [97]. So we can say that our book will cover these two books largely. Let me also mention that our book will cover [2]. Our book will also cover or overlap [4, 7] in some sense. However, the approach will be quite different. For example, I did my best to explain how we must be careful when we use the complex interpolation functor defined in [7]. Theorem 2.34 appeared in [100]. The atomic decomposition appeared in [23, 101]. Although there is strong overlap with these three books in our book, I did not describe the compactness of the embedding in depth, unlike [23, 101]. Its variant, quarkonial decomposition, appeared in [103, 104]. In the context of the atomic decomposition, I have included quarkonial decomposition. However, I present a different application of the quarkonial decomposition from [103, 104].

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My earlier manuscript of 2005 was not so polished, and because of this my presentation of the seminar was poor. I apologize here for confusing the participants of my seminar and thank them for their attendance.

Notation in This Book

Sets and Set Functions

1. The metric open ball defined by ℓ^2 is usually called a *ball*. We denote by $B(x, r)$ the *ball centered at x of radius r*. Namely, we write

$$B(x, r) \equiv \{y \in \mathbb{R}^n \ : \ \|x - y\| < r\}$$

when $x \in \mathbb{R}^n$ and $r > 0$. Given a ball B, we denote by $c(B)$ its *center* and by $r(B)$ its *radius*. We write $B(r)$ instead of $B(o, r)$, where $o \equiv (0, 0, \ldots, 0)$.

2. By a "cube" we mean a compact cube whose edges are parallel to the coordinate *axes*. The metric closed ball defined by ℓ^∞ is called a *cube*. If a cube has center x and radius r, we denote it by $Q(x, r)$. Namely, we write

$$Q(x, r) \equiv \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \ : \ \max_{j=1,2,\ldots,n} |x_j - y_j| \leq r \right\}$$

when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. From the definition of $Q(x, r)$, its volume is $(2r)^n$. We write $Q(r)$ instead of $Q(o, r)$. Given a cube Q, we denote by $c(Q)$ the *center of Q* and by $\ell(Q)$ the *sidelength of Q*: $\ell(Q) = |Q|^{1/n}$, where $|Q|$ denotes the volume of the cube Q.

3. Given a cube Q and $k > 0$, $k\,Q$ means the *cube concentric to Q with sidelength $k\,\ell(Q)$*. Given a ball B and $k > 0$, we denote by $k\,B$ the *ball concentric to B with radius $k\,r(B)$*.

4. For $\nu \in \mathbb{Z}$ and $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, we define $Q_{\nu m} \equiv \prod_{j=1}^{n} \left[\frac{m_j}{2^\nu}, \frac{m_j + 1}{2^\nu}\right)$. Denote by $\mathscr{D} = \mathscr{D}(\mathbb{R}^n)$ the set of such cubes. The elements in $\mathscr{D}$ are called dyadic cubes.

5. Let E be a measurable set. Then we denote its *indicator function* by χ_E. If E has positive measure and E is integrable over f, then denote by $m_E(f)$ the *average of f* over E. $|E|$ denotes the *volume of E*.

6. If we are working on $\mathbb{R}^n$, then $\mathscr{B}$ denotes the set of all balls in $\mathbb{R}^n$, while $\mathscr{Q}$ denotes the set of all cubes in $\mathbb{R}^n$. Be careful because $\mathscr{B}$ can be used for a different purpose: When we are working on a measure space $(X, \mathscr{B}, \mu)$, then $\mathscr{B}$ stands for the set of all Borel sets.
7. The symbol $\sharp A$ means the *cardinality of the set* A.
8. The symbol 2^X denotes the set of all subsets in X.
9. Let X be a topological space. Then $\mathscr{K}_X$ is the set of all compact subsets of X, and $\mathscr{O}_X$ is the set of all open subsets of X.
10. The set $\mathscr{I}(\mathbb{R})$ denotes the set of all closed intervals in $\mathbb{R}$.
11. A tacit understanding is that by a "cube" we mean a closed cube whose edges are parallel to the coordinate axes. However, we say that dyadic cubes are also cubes.
12. We define the *upper half space* $\mathbb{R}^n_+$ and the *lower half space* $\mathbb{R}^n_-$ by

$$\mathbb{R}^n_\pm \equiv \{(x', x_n) \in \mathbb{R}^n \,:\, \pm x_n > 0\}. \tag{1}$$

Numbers

1. Let $a \in \mathbb{R}$. Then write $a_+ = a \vee 0 \equiv \max(a, 0)$ and $a_- = a \wedge 0 \equiv \min(a, 0)$. Correspondingly, given an $\mathbb{R}$-valued function f, f_+ and f_- are functions given by $f_+(x) \equiv \max(f(x), 0)$ and $f_-(x) \equiv \min(f(x), 0)$, respectively.
2. Let $a, b \in \mathbb{R}$. Then write $a \vee b \equiv \max(a, b)$ and $a \wedge b \equiv \min(a, b)$. Correspondingly, given $\mathbb{R}$-valued functions f, g, $f \vee g$ and $f \wedge g$ are functions given by $f \vee g(x) \equiv \max(f(x), g(x))$ and $f \wedge g(x) \equiv \min(f(x), g(x))$, respectively.
3. The constants C and c denote positive constants that may change from one occurrence to another. Because the two constants c can be different, the inequality $0 < 2c < c$ is by no means a contradiction. When we add a subscript, for example, this means that the constant c depends upon the parameter. It can happen that the constants with subscript differ according to the above rule. In particular, we prefer to use c_n for various constants that depend on n, when we do not want to specify its precise value.
4. Let $A, B \geq 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \leq CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously, while $A \simeq B$ means that there exists a constant $C > 0$ such that $A = CB$.
5. When we need to emphasize or keep in mind that the constant C depends on the parameters α, β, γ, etc:

   (a) Instead of $A \lesssim B$, we write $A \lesssim_{\alpha,\beta,\gamma,\ldots} B$.
   (b) Instead of $A \gtrsim B$, we write $A \gtrsim_{\alpha,\beta,\gamma,\ldots} B$.
   (c) Instead of $A \sim B$, we write $A \sim_{\alpha,\beta,\gamma,\ldots} B$.
   (d) Instead of $A \simeq B$, we write $A \simeq_{\alpha,\beta,\gamma,\ldots} B$.

6. We define

$$\mathbb{N} \equiv \{1, 2, \ldots\}, \quad \mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}, \quad \mathbb{N}_0 \equiv \{0, 1, \ldots\}.$$

7. We denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$, the coefficient field under consideration.
8. For $a \in \mathbb{R}^n$, we write $\langle a \rangle \equiv \sqrt{1 + |a|^2}$.
9. When $0 < p, q \le \infty$, we define $\sigma_p \equiv n \left(\frac{1}{p} - 1\right)_+$, $\sigma_{p,q} \equiv \max(\sigma_p, \sigma_q)$.
10. We sometimes identify $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$.

Function Spaces

1. We use $\star$ for functions; $f = f(\star)$.
2. The function spaces are tacitly on $\mathbb{R}^n$. But sometimes, we will work on an open set Ω with C^∞-boundary.
3. Let X be a Banach space. We denote its norm by $\|\star\|_X$. However, we sometimes denote the $L^p(\mathbb{R}^n)$-norm by $\| \star \|_p$.
4. Let Ω be an open set in $\mathbb{R}^n$. Then $C_{\rm c}^\infty(\Omega)$ denotes the set of smooth functions with compact support in Ω.
5. Let $1 \le j \le n$. The symbol x_j denotes not only the j-th coordinate but also the function $x = (x_1, \ldots, x_n) \mapsto x_j$.
6. Suppose that $\{f_j\}_{j=1}^\infty$ is a sequence of measurable functions. Then we write

$$\|f_j\|_{L^p(\ell^q)} \equiv \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{p}{q}} {\rm d}x \right)^{\frac{1}{p}} \quad (0 < p, q \le \infty)$$

and

$$\|f_j\|_{\ell^q(L^p)} \equiv \left(\sum_{j=1}^\infty \left(\int_{\mathbb{R}^n} |f_j(x)|^p {\rm d}x \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \quad (0 < p, q \le \infty).$$

7. The space $L^2(\mathbb{R}^n)$ is the Hilbert space of square integrable functions on $\mathbb{R}^n$ whose inner product is given by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g}(x){\rm d}x. \tag{2}$$

8. In view of (2), the inner product of $L^2(\mathbb{R}^n)$, it seems appropriate that we define the embedding $L^1_{\rm loc}(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n)$ by

$$f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n) \mapsto F_f \equiv \left[g \in \mathscr{S}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} g(x)\overline{f(x)}{\rm d}x \right].$$

However, in order that $f \mapsto F_f$ be linear, we will define it later by

$$f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n) \mapsto F_f \equiv \left[g \in \mathscr{S}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} g(x)f(x){\rm d}x \right].$$

9. Let E be a measurable set and f be a measurable function with respect to the Lebesgue measure. Then write $m_E(f) \equiv \dfrac{1}{|E|}\displaystyle\int_E f$.
10. Let $0 < \eta < \infty$, E be a measurable set, and f be a positive measurable function with respect to the Lebesgue measure. Then write $m_E^{(\eta)}(f) \equiv m_E(f^\eta)^{\frac{1}{\eta}}$.
11. Let $0 < \eta < \infty$. We define the *powered Hardy–Littlewood maximal operator* $M^{(\eta)}$ by

$$M^{(\eta)}f(x) \equiv \sup_{R>0}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)}|f(y)|^\eta{\rm d}y\right)^{\frac{1}{\eta}}.$$

12. For $x \in \mathbb{R}^n$, we define $\langle x \rangle \equiv \sqrt{1+|x|^2}$.
13. The space C denotes the set of all continuous functions on $\mathbb{R}^n$.
14. The space ${\rm BC}(\mathbb{R}^n)$ denotes the set of all bounded continuous functions on $\mathbb{R}^n$.
15. The space ${\rm BUC}(\mathbb{R}^n)$ denotes the set of all bounded uniformly continuous functions on $\mathbb{R}^n$.
16. Occasionally we identify the value of functions with functions. For example, $\sin x$ denotes the function on $\mathbb{R}$ defined by $x \mapsto \sin x$.
17. Given a Banach space X, we denote by X^* its dual space. The set X_1 is the closed unit ball in X.
18. Let μ be a measure on a measure space $(X, \mathscr{B}, \mu)$. Given a μ-measurable set A with positive μ-measure and a function f, we write

$$m_Q(f) \equiv \frac{1}{\mu(A)}\int_A f(x){\rm d}\mu(x).$$

Let $0 < \eta < \infty$. Then define $m_Q^{(\eta)}(f) \equiv m_Q(f^\eta)^{\frac{1}{\eta}}$ whenever f is positive.

19. For $x \in \mathbb{R}^n$, we define $\mathscr{Q}_x$ to be the set of all cubes containing x. Given a measurable function, Mf denotes the uncentered Hardy–Littlewood maximal operator and $M'f$ denotes the centered Hardy–Littlewood maximal operator.

$$Mf(x) \equiv \sup_{Q \in \mathscr{Q}_x} m_Q(|f|),$$

$$M'f(x) \equiv \sup_{r>0} m_{Q(x,r)}(|f|).$$

We sometimes use balls instead of cubes.

20. If notational confusion seems likely, Then we use [ ] to denote $Mf(x) = M[f](x)$, $\mathscr{F}\varphi(\xi) = \mathscr{F}[\varphi](\xi)$, etc.
21. If notational confusion seems likely, for the Fourier transform $\mathscr{F}$ and the Hardy–Littlewood maximal operator M, we use [ ] to denote $\mathscr{F}[f + g + h]$, $M[f + g + h]$.
22. For $j = 1, 2, \ldots, n$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$ stands for the partial derivative. The symbol Δ stands for the Laplacian $\sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}$.
23. We denote the $L^p(\mathbb{R}^n)$-norm by $\| \star \|_p$. For other function spaces such as the Hölder continuous function space $\mathscr{C}^s(\mathbb{R}^n)$, we use $\| \star \|_{\mathscr{C}^s}$ to stress the function spaces.
24. When we consider function spaces on a domain Ω, we denote by $\mathscr{C}^s(\Omega)$ the *Hölder continuous function space of order* s.
25. A quasi-norm over a linear space X enjoys positivity, homogeneity, and quasi-triangle inequality: for some $\alpha \geq 1$, $\|f + g\|_X \leq \alpha(\|f\|_X + \|g\|_X) \quad (f, g \in X)$. However, to simplify, we frequently omit the word "quasi". Likewise we abbreviate the word "quasi-Banach space" to Banach space.
26. Let $j \in \mathbb{Z}$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then define $\varphi_j \equiv \varphi(2^{-j}\star)$ and $\varphi^j \equiv 2^{jn}\varphi(2^j\star)$.
27. The Kronecker delta function is given by $\delta_{jk} \equiv \begin{cases} 1 & (j = k), \\ 0 & (j \neq k). \end{cases}$ for $j, k \in \mathbb{Z}$.
28. When two normed space X, Y are isomorphic, we write $X \approx Y$.
29. For subsets A, B of a linear space V and $v \in V$, define the Minkovski sum by

$$v + A \equiv \{v + a \ : \ a \in A\}, \quad A + B \equiv \{a + b \ : \ a \in A, \ b \in B\}.$$

30. When two topological spaces X, Y are homeomorphic, we write $X \approx Y$.
31. When A and B are sets, $A \subset B$ stands for the inclusion of sets. If, in addition, both A and B are topological spaces, and if the natural embedding mapping $A \to B$ is continuous, we write $A \hookrightarrow B$ in the sense of continuous embedding.

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