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Introduction to Stochastic Calculus

Springer

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Dedicated to the memory of G. Kallianpur

Preface

This book is a comprehensive textbook on stochastic calculus—the branch of mathematics that is most widely applied in financial engineering and mathematical finance. It will be useful for a two-semester graduate-level course on stochastic calculus, where the background required is a course on measure-theoretic probability.

This book begins with conditional expectation and martingales, and basic results on martingales are included with proofs (in discrete time as well as in continuous time). Then a chapter on Brownian motion and Ito's integration with respect to the Brownian motion follows, which includes stochastic differential equations. These three chapters give a soft landing to a reader to the more complex results that follow. The first three chapters form the introductory material.

Taking a cue from the Ito's integral, a *stochastic integrator* is defined and its properties, as well as the properties of the integral, are discussed. In most treatments, one starts by defining the integral for a square integrable martingale and where integrands themselves are in suitable Hilbert space. Then over several stages, the integral is extended, and at each step, one has to reaffirm its properties. We avoid this. Various results including quadratic variation and Ito's formula follow from the definition. Then Emery topology is defined and studied.

We then show that for a square integrable martingale M, the quadratic variation $[M,M]$ exists, and using this, we show that square integrable martingales are stochastic integrators. This approach to stochastic integration is different from the standard approach as we do not use Doob–Meyer decomposition. Instead of using the predictable quadratic variation $\langle M,M\rangle$ of a square integrable martingale M, we use the quadratic variation $[M,M]$. Using an inequality by Burkholder, we show that all martingales and local martingales are stochastic integrators and thus semimartingales are stochastic integrators. We then show that stochastic integrators are semimartingales and obtain various results such as a description of the class of integrands for the stochastic integral. We complete the chapter by giving a proof of the Bichteler–Dellacherie–Meyer–Mokobodzky theorem.

These two chapters form the basic material. We have avoided invoking results from functional analysis but rather included the required steps. Thus, instead of saying that the integral is a continuous linear functional on a dense subset of a Banach space and hence can be extended to the Banach space, we explicitly construct the extension.

Next, we introduce *Pathwise formulae* for the quadratic variation and the stochastic integral. These have not found a place in any textbook on stochastic integration. We briefly specialize in continuous semimartingales and obtain growth estimates and study the solution of a stochastic differential equation (SDE) using the technique of random time change. We also prove pathwise formulae for the solution of an SDE driven by continuous semimartingales.

Then, we move on to a study of predictable increasing processes, introduce predictable stopping times and prove the Doob–Meyer decomposition theorem.

The Davis inequality ($p = 1$ case of the Burkholder–Davis–Gundy inequality) plays an important role in the integral representation of martingales and hence is taken up next. We also introduce the notion of a sigma-martingale.

We then give a comprehensive treatment of integral representation of martingales and its connection with the uniqueness of equivalent martingale measure. This connection is important from the point of view of mathematical finance. Here, we consider the multivariate case and also include the case when the underlying process is a sigma-martingale.

In order to study stochastic differential equations driven by a general semimartingale, we introduce the Metivier–Pellaumail inequality and, using it, introduce a notion of the dominating process of a semimartingale. We then obtain existence and uniqueness of solutions to the SDE and also obtain a pathwise formula by showing that modified Euler–Peano approximations converge almost surely.

We conclude this book by discussing the Girsanov theorem and its role in the construction of weak solutions to SDEs.

We would like to add that this book includes various techniques that we have learnt over the last four decades from different sources. This includes, in addition to books and articles given in the references, the *Séminaire de Probabilités* volumes and various books on stochastic processes and research articles. We must also mention the blog https://almostsure.wordpress.com/stochastic-calculus/ by George Lowther, which brought some of these techniques to our attention.

Siruseri, India Rajeeva L. Karandikar
B. V. Rao

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