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Ramji Lal

Algebra 1

Groups, Rings, Fields and Arithmetic

Springer

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Dedicated to the memory of
my mother
(Late) Smt Murti Devi,
my father
(Late) Sri Sankatha Prasad Lal, and
my father-like brother
(Late) Sri Gopal Lal

Preface

Algebra has played a central and decisive role in all branches of mathematics and, in turn, in all branches of science and engineering. It is not possible for a lecturer to cover, physically in a classroom, the amount of algebra which a graduate student (irrespective of the branch of science, engineering, or mathematics in which he prefers to specialize) needs to master. In addition, there are a variety of students in a class. Some of them grasp the material very fast and do not need much of assistance. At the same time, there are serious students who can do equally well by putting a little more effort. They need some more illustrations and also more exercises to develop their skill and confidence in the subject by solving problems on their own. Again, it is not possible for a lecturer to do sufficiently many illustrations and exercises in the classroom for the purpose. This is one of the considerations which prompted me to write a series of three volumes on the subject starting from the undergraduate level to the advance postgraduate level. Each volume is sufficiently rich with illustrations and examples together with numerous exercises. These volumes also cater for the need of the talented students with difficult, challenging, and motivating exercises which were responsible for the further developments in mathematics. Occasionally, the exercises demonstrating the applications in different disciplines are also included. The books may also act as a guide to teachers giving the courses. The researchers working in the field may also find it useful.

The present (first) volume consists of 11 chapters which starts with language of mathematics (logic and set theory) and centers around the introduction to basic algebraic structures, viz. group, rings, polynomial rings, and fields, together with fundamentals in arithmetic. At the end of this volume, there is an appendix on the basics of category theory. This volume serves as a basic text for the first-year course in algebra at the undergraduate level. Since this is the first introduction to the abstract-algebraic structures, we proceed rather leisurely in this volume as compared with the other volumes.

The second volume contains ten chapters which includes the fundamentals of linear algebra, structure theory of fields and Galois theory, representation theory of finite groups, and the theory of group extensions. It is needless to say that linear

algebra is the most applicable branch of mathematics and it is essential for students of any discipline to develop expertise in the same. As such, linear algebra is an integral part of the syllabus at the undergraduate level. General linear algebra, Galois theory, representation theory of groups, and the theory of group extensions follow linear algebra which is a part, and indeed, these are parts of syllabus for the second- and third-year students of most of the universities. As such, this volume may serve as a basic text for second- and third-year courses in algebra.

The third volume of the book also contains 10 chapters, and it can act as a text for graduate and advanced postgraduate students specializing in mathematics. This includes commutative algebra, basics in algebraic geometry, homological methods, semisimple Lie algebra, and Chevalley groups. The table of contents gives an idea of the subject matter covered in the book.

There is no prerequisite essential for the book except, occasionally, in some illustrations and starred exercises, some amount of calculus, geometry, or topology may be needed. An attempt to follow the logical ordering has been made throughout the book.

My teacher (Late) Prof. B.L. Sharma, my colleague at the University of Allahabad, my friend Dr. H.S. Tripathi, my students Prof. R.P. Shukla, Prof. Shivdatt, Dr. Brajesh Kumar Sharma, Mr. Swapnil Srivastava, Dr. Akhilesh Yadav, Dr. Vivek Jain, Dr. Vipul Kakkar, and above all the mathematics students of the University of Allahabad had always been the motivating force for me to write a book. Without their continuous insistence, it would have not come in the present form. I wish to express my warmest thanks to all of them.

Harish-Chandra Research Institute (HRI), Allahabad, has always been a great source for me to learn more and more mathematics. I wish to express my deep sense of appreciation and thanks to HRI for providing me all the infrastructural facilities to write these volumes.

Last but not least, I wish to express my thanks to my wife Veena Srivastava who had always been helpful in this endeavor.

In spite of all care, some mistakes and misprint might have crept in and escaped my attention. I shall be grateful to any such attention. Criticisms and suggestions for the improvement of the book will be appreciated and gratefully acknowledged.

Allahabad, India Ramji Lal
April 2017

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About the Author

Ramji Lal is Adjunct Professor at the Harish-Chandra Research Institute (HRI), Allahabad, Uttar Pradesh. He started his research career at the Tata Institute of Fundamental Research (TIFR), Mumbai, and served at the University of Allahabad in different capacities for over 43 years: as a Professor, Head of the Department, and the Coordinator of the DSA program. He was associated with HRI, where he initiated a postgraduate (PG) program in mathematics and Coordinated the Nurture Program of National Board for Higher Mathematics (NBHM) from 1996 to 2000. After his retirement from the University of Allahabad, he was an Advisor cum Adjunct Professor at the Indian Institute of Information Technology (IIIT), Allahabad, for over 3 years. His areas of interest include group theory, algebraic K-theory, and representation theory.

Notations from Algebra 1

$\langle a \rangle$	Cyclic subgroup generated by a, p. 122
a/b	a divides b, p. 57
$a \sim b$	a is an associate of b, p. 57
A^t	The transpose of a matrix A, p. 201
$A^\star$	The hermitian conjugate of a matrix A, p. 215
$Aut(G)$	The automorphism group of G, p. 103
A_n	The alternating group of degree n, p. 175
$B(n, \mathbb{R})$	Borel subgroup, p. 189
$C_G(H)$	The centralizer of H in G, p. 160
$\mathbb{C}$	The field of complex numbers, p. 78
D_n	The dihedral group of order $2n$, p. 90
det	Determinant map, p. 193
$End(G)$	Semigroup of endomorphisms of G, p. 103
$f(A)$	Image of A under the map f, p. 33
$f^{-1}(B)$	Inverse image of B under the map f, p. 33
$f\|Y$	Restriction of the map f to Y, p. 29
E_{ij}^{λ}	Transvections, p. 201
$Fit(G)$	Fitting subgroup, p. 357
$g.c.d.$	Greatest common divisor, p. 58
$g.l.b.$	Greatest lower bound, or inf, p. 39
$G/^l H (G/^r H)$	The set of left(right) cosets of G mod H, p. 133
G/H	The quotient group of G modulo H, p. 150
$[G : H]$	The index of H in G, p. 133
$\|G\|$	Order of G
$G' = [G, G]$	Commutator subgroup of G
G^n	nth term of the derived series of G, p. 348
$GL(n, \mathbb{R})$	General linear group, p. 187
I_X	Identity map on X, p. 29
i_Y	Inclusion map from Y, p. 30
$Inn(G)$	The group of inner automorphisms

$ker\, f$	The kernel of the map f, p. 35
$L_n(G)$	nth term of the lower central series of G
l.c.m.	Least common multiple, p. 58
l.u.b.	Least upper bound, or sup, p. 39
$M_n(R)$	The ring of $n \times n$ matrices with entries in R
$\mathbb{N}$	Natural number system, p. 22
$N_G(H)$	Normalizer of H in G, p. 160
$O(n)$	Orthogonal group, p. 198
$O(1, n)$	Lorentz orthogonal group, p. 202
$PSO(1, n)$	Positive special Lorentz orthogonal group, p. 203
$\mathbb{Q}$	The field of rational numbers, p. 73
Q_8	The Quaternion group, p. 88
$\mathbb{R}$	The field of real numbers, p. 75
$R(G)$	Radical of G, p. 349
S_n	Symmetric group of degree n, p. 88
$Sym(X)$	Symmetric group on X, p. 88
S^3	The group of unit Quaternions, p. 91
$\langle S \rangle$	Subgroup generated by a subset S, p. 116
$SL(n, \mathbb{R})$	Special linear group, p. 196
$SO(n)$	Special orthogonal group, p. 199
$SO(1, n)$	Special Lorentz orthogonal group, p. 203
$SP(2n, \mathbb{R})$	Symplectic group, p. 202
$SU(n)$	Special unitary group, p. 204
$U(n)$	Unitary group, p. 204
U_m	Group of prime residue classes modulo m, p. 99
V_4	Klein's four group, p. 101
X/R	The quotient set of X modulo R, p. 36
R_x	Equivalence class modulo R determined by x, p. 27
X^+	Successor of X, p. 20
X^Y	The set of maps from Y to X, p. 33
$\subset$	Proper subset, p. 15
$\wp(X)$	Power set of X, p. 19
$\prod_{k=1}^{n} G_k$	Direct product of groups $G_k, 1 \leq k \leq n$, p. 142
$\trianglelefteq$	Normal subgroup, p. 148
$\trianglelefteq\trianglelefteq$	Subnormal subgroup, p. 335
$Z(G)$	Center of G, p. 112
$\mathbb{Z}_m$	The ring of residue classes modulo m, p. 80
$p(n)$	The number of partition of n, p. 209
$H \prec K$	Semidirect product of H with K, p. 206
$\sqrt{A}$	Radical of an ideal A, p. 233
$R(G)$	Semigroup ring of a ring R over a semigroup G, p. 239
$R[X]$	Polynomial ring over the ring R in one variable, p. 241
$R[X_1, X_2, \cdots, X_n]$	Polynomial ring in several variables, p. 249
μ	The Mobius function, p. 257

σ	Sum of divisor function, p. 257
$\left(\frac{a}{p}\right)$	Legendre symbol, p. 282
$Stab(G, X)$	Stabilizer of an action of G on X, p. 298
G_x	Isotropy subgroup of an action of G at x, p. 298
X^G	Fixed point of an action of G on X
$Z_n(G)$	nth term of the upper central series of G, p. 354
$\Phi(G)$	The Frattini subgroup of G, p. 358