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Continued Fractions

Second edition

Volume 1: Convergence Theory

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Preface to the second edition

15 years have passed since the first edition of this book was written. A lot has happened since then – also in continued fraction theory. New ideas have emerged and some old results have gotten new proofs. It was therefore time to revise our book “Continued Fractions with Applications” which appeared in 1992 on Elsevier.

The interest in using continued fractions to approximate special functions has also grown since then. Such fractions are easy to program, they have impressive convergence properties, and their convergence is often easy to accelerate. They even have good and reliable truncation error bounds which makes it possible to control the accuracy of the approximation. The bounds are both of the *a posteriori type* which tells the accuracy of a done calculation, and of the *a priori type* which can be used to determine the number of terms needed for a wanted accuracy. This important aspect is treated in this first volume of the second edition, along with the basic theory.

In the second volume we focus more on continued fraction expansions of analytic functions. There are several beautiful connections between analytic function theory and continued fraction expansions. We can for instance mention orthogonal polynomials, moment theory and Padé approximation.

We have tried to give credit to people who have contributed to the continued fraction theory up through the ages. But some of the material we believe to be new, at least we have found no counterpart in the literature. In particular, we believe that tail sequences play a more fundamental role in this book than what is usual. This way of looking at continued fractions is very fruitful.

Each chapter is still followed by a number of problems. This time we have marked the more theoretic ones by ♠. We have also kept the appendix from the first edition. This list of continued fraction expansions of special functions was so well received that we wanted it to stay as part of the book. Finally, we have kept the informality in the sense that the first chapter consists almost entirely of examples which show what continued fractions are good for. The more serious theory starts in Chapter 2.

Lisa Lorentzen carries the main responsibility for the revisions in this second edition. Through the first year of its making, Haakon Waadeland was busy writing a handbook on continued fractions, together with an international group of people. This left Lisa Lorentzen with quite free hands to choose the contents and the way of presentation. Still, he has played an important part in the later phases of the work on volume 1. For volume 2 Lisa Lorentzen bears the blame alone.

Trondheim, 14 February 2008

Lisa Lorentzen

Haakon Waadeland

Preface

The name

Shortly before this book was finished, we sent out a number of copies of Chapter 1, under the name “A Taste of Continued Fractions”. Now, in the process of working our way through the chapters on a last minute search for errors, unintended omissions and overlaps, or other unfortunate occurrences, we feel that this title might have been the right one even for the whole book. In most of the chapters, in particular in the applications, a lot of work has been put into the process of cutting, canceling and “non-writing”. In many cases we are just left with a “taste”, or rather a glimpse of the role of the continued fractions within the topic of the chapter. We hope that we thereby can open some doors, but in most cases we are definitely not touring the rooms.

The chapters

Each chapter starts with some introductory information, “About this chapter”. The purpose is not to tell about the contents in detail. That has been done elsewhere. What we want is to tell about the intention of the chapter, and thereby also to adjust the expectations to the right (moderate) level. Each chapter ends with a reference list, reflecting essentially literature used in preparing that particular chapter. As a result, books and papers will in many cases be referred to more than once in the book. On the other hand, those who look for a complete, updated bibliography on the field will look in vain. To present such a bibliography has not been one of the purposes of the book.

The authors

The two authors are different in style and approach. We have not made an effort to hide this, but to a certain extent the creative process of tearing up each other’s drafts and telling him/her to glue it together in a better way (with additions and omissions) may have had a certain disguising effect on the differences. This struggling type of cooperation leaves us with a joint responsibility for the whole book. The way we then distribute blame and credit between us is an internal matter.

The treasure chest

Anybody who has lived with and loved continued fractions for a long time will also have lived with and loved the monographs by Perron, Wall and Jones/Thron. Actually the love for continued fractions most likely has been initiated by one or more of these books. This is at least the case for the authors of the present book, and more so: these three books have played an essential role in our lives. The present book is in no way an attempt to replace or compete with these books. To the contrary, we hope to urge the reader to go on to these sources for further information.

For whom?

We are aiming at two kinds of readers: On the one hand people *in or near* mathematics, who are curious about continued fractions; on the other hand senior-graduate level students who would like an introduction (and a little more) to the analytic theory of continued fractions. Some basic knowledge about functions of a complex variable, a little linear algebra, elementary differential equations and occasionally a little dash of measure theory is what is needed of mathematical background. Hopefully the students will appreciate the problems included and the examples. They may even appreciate that some examples precede a properly established theory. (Others may dislike it.)

Words of gratitude

We both owe a lot to Wolf Thron, for what we have learned from him, for inspiration and help, and for personal friendship. He has read most of this book, and his remarks, perhaps most of all his objections, have been of great help for us. Our gratitude also extends to Bill Jones, his closest coworker, to Arne Magnus, whose recent death struck us with sadness, and to all other members of the Colorado continued fraction community. Here in Trondheim Olav Njåstad has been a key person in the field, and we have on several occasions had a rewarding cooperation with him.

Many people, who had received our Chapter I, responded by sending friendly and encouraging letters, often with valuable suggestions. We thank them all for their interest and kind help.

The main person in the process of changing the hand-written drafts to a camera-ready copy was Leiv Arild Andenes Jacobsen. His able mastering of LaTeX, in combination with hard work, often at times when most people were in bed, has left us with a great debt of gratitude. We also want to thank Arild Skjølsvold and Irene Jacobsen for their part of the typing job. We finally thank Ruth Waadeland, who made all the drawings, except the LaTeX-made ones in Chapter XI.

The Department of Mathematics and Statistics, AVH, The University of Trondheim generously covered most of the typing expenses. The rest was covered by Elsevier Science Publishers. We are most grateful to Claude Brezinski and Luc Wuytack for urging us to write this book, and to Elsevier Science Publishers for publishing it.

Trondheim, December 1991,

Lisa Lorentzen

Haakon Waadeland

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