ATLANTIS STUDIES IN MATHEMATICS FOR ENGINEERING AND SCIENCE

VOLUME 7

SERIES EDITOR: C.K. CHUI

Atlantis Studies in Mathematics for Engineering and Science

Series Editor:

C. K. Chui, Stanford University, USA

(ISSN: 1875-7642)

Aims and scope of the series

The series 'Atlantis Studies in Mathematics for Engineering and Science' (AMES) publishes high quality monographs in applied mathematics, computational mathematics, and statistics that have the potential to make a significant impact on the advancement of engineering and science on the one hand, and economics and commerce on the other. We welcome submission of book proposals and manuscripts from mathematical scientists worldwide who share our vision of mathematics as the engine of progress in the disciplines mentioned above.

All books in this series are co-published with World Scientific.

For more information on this series and our other book series, please visit our website at:

www.atlantis-press.com/publications/books

AMSTERDAM – PARIS

© ATLANTIS PRESS / WORLD SCIENTIFIC

Boundary Element Methods with Applications to Nonlinear Problems

Goong Chen

Professor of Mathematics and Aerospace Engineering, Texas A&M
University

College Station, Texas 77843, USA

and

Distinguished Professor of Mathematics

National Taiwan University

Taipei, Taiwan, Republic of China

Jianxin Zhou

Professor of Mathematics

Texas A&M University

College Station, Texas 77843, USA

**ATLANTIS
PRESS**

AMSTERDAM – PARIS

World Scientific

Atlantis Press

29, avenue Laumière
75019 Paris, France

For information on all Atlantis Press publications, visit our website at: www.atlantis-press.com

Copyright

This book, or any parts thereof, may not be reproduced for commercial purposes in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system known or to be invented, without prior permission from the Publisher.

Atlantis Studies in Mathematics for Engineering and Science

Volume 1: Continued Fractions: Volume 1: Convergence Theory – L. Lorentzen, H. Waadeland

Volume 2: Mean Field Theories and Dual Variation – T. Suzuki

Volume 3: The Hybrid Grand Unified Theory – V. Lakshmikantham, E. Escultura, S. Leela

Volume 4: The Wavelet Transform – R.S. Pathak

Volume 5: Theory of Causal Differential Equations – V. Lakshmikantham, S. Leela, Z. Drici

Volume 6: The Omega Problem of all Members of the United Nations – E.N. Chukwu

ISBN: 978-90-78677-31-4
ISSN: 1875-7642

e-ISBN: 978-94-91216-27-5

Dedicated to
Professor David L. Russell
on the occasion of his 70th birthday
and

Professor George C. Hsiao
on the occasion of his 75th birthday

Preface to the 1st edition

Boundary element methods (BEM) have undergone rapid advancement in recent years. Because of their numerous advantages, such as ease of coding, small memory requirement and computational efficiency, they have become a *major* numerical tool. The analysis and rigor of BEM have been strengthened through the work of several mathematicians. It is now possible for the authors to fit together the many scattered contributions into a comprehensive account. In this book, we present mathematical formulations of *boundary integral equations* (BIE) for several of the most important linear elliptic *boundary value problems* (BVP), discuss their computational algorithms and the accuracy of their solutions, illustrate the numerical solutions and show some applications. We wrote this monograph as a reference source for researchers who are concerned with numerical solutions of *partial differential equations* (PDE), and as a graduate text for a course in this subject.

It has been known for a long time that it is possible to formulate BIE with singular kernels to solve BVP of the linear elliptic type. The unknown boundary data in the solution depend on the prescribed boundary data through those BIE. Once all the boundary data (or the Cauchy data) become available, the solution of the given PDE is obtained by an application of the Green's formula. Complex variable techniques for solving BIE on planar complex contours were perfected by the Russian school (see Muskhelishvili [137]) in the early 1950s, and are still widely used today for *two-dimensional* potential and elasticity problems. A monumental achievement, developing a *general theory* of BIE using the *fundamental solution* rather than the complex Cauchy kernel, is presented in V.D. Kupradze's book [117], published in 1965. In that decade, with electronic computers gaining larger power and faster speed, and being in wider use in the U.S., T.A. Cruse, F.J. Rizzo [58, 160, 59] and others started to compute solutions to many BIE arising in continuum mechanics and obtained excellent numerical results. The BIE they used at that time were primarily formulated from the Green's formula, which is commonly referred to as *the direct approach* in the literature.

Actually, for elliptic BVP an alternative way of formulating simpler systems of BIE is possible. In a paper [72] published in 1961 G. Fichera used ideas from *potential theory* to represent the solution as a *simple-layer potential*. This *ansatz* leads to greatly simplified systems of BIE whose solutions, as a consequence, are much easier to compute. Nevertheless, Fichera's paper remained largely obscure until more than a decade later, when in 1973 G.C. Hsiao and R.C. MacCamy [95] adapted his ideas and lucidly exemplified them for the Poisson equation on an exterior domain, and for the second biharmonic problem in two-dimensional linear elasticity. The arguments in [95] are now standard in proving the existence and uniqueness of solutions of BIE, as are shown at several places in our book.

When BIE are approximated numerically, the questions of *convergence* and *error analysis* need to be addressed. The mathematical foundation of such an analysis was laid down in 1977 in a paper [96] by G.C. Hsiao and W.L. Wendland. There, they used the *strong ellipticity* of the simple-layer boundary operator to derive sharp error estimates for the solutions of the *Galerkin approximations* of BIE. Their proof works in *arbitrary space dimensions* for any strongly elliptic pseudodifferential equations arising from elliptic BVP.

In practice, however, BIE are more easily approximated by *point collocation*. Error analysis questions concerning collocation approximations were essentially resolved by D.N. Arnold, J. Saranen and W.L. Wendland in [9, 165, 10] in the case of *two-dimensional* problems, where in the culminating paper [10] the main technique was a delicate Fourier series analysis. Nevertheless, as of this writing, no analogous success has been achieved for collocation analysis of three-dimensional problems.

The authors' interests in BEM stem from numerical solutions to various applied problems in science and engineering. In our pursuit of numerical solutions of PDE, we marvelled at the efficacy of the BEM and we were gratified by the accuracy of the numerical results. During those years we have also received growing demands from graduate students in many engineering departments asking us to teach courses containing BEM material. While teaching those courses, we began to realize the needs of developing a graduate level text and monograph that is sufficiently *self-contained*, emphasizes more *mathematical rigor*, and includes many useful *examples of applications and illustrations* to stimulate the audience's interests. Indeed, these are the three major goals we have been striving to achieve throughout the volume—we hope with some success.

The organization and selection of the material carry the strong bias of the authors. There are 10 chapters in the book. The first five consist of preparatory material:

- (1) Chapter 1, an introduction, gives readers some quick ideas as how BEM work and how they compare with other numerical PDE methods.
- (2) Sobolev spaces are fundamental in the modern theory of PDE. In Chapter 2, we explain some of their basic properties that are essential in understanding the solvability of BIE.
- (3) Hypersingular integrals occur naturally in boundary element computations. The main objective of Chapter 3 is to enable readers to understand how to regularize such integrals based upon the theory of distributions.
- (4) Boundary integral operators can be studied in an elegant way using the theory of pseudodifferential and Fredholm operators. In Chapter 4, we give a concise account of this approach and its applications to BIE.
- (5) The mathematical theory of finite elements is a prerequisite for BEM. In Chapter 5, we describe the commonly used finite element spaces and their basic properties.

The subsequent chapters deal with the corpus of BEM for PDE:

- (6) Chapter 6 studies BIE for the potential equation.
- (7) Chapter 7 studies BIE for the Helmholtz equation.
- (8) Chapter 8 studies BIE for the thin plate equation.
- (9) Chapter 9 studies BIE for the linear elastostatic systems.
- (10) Finally, Chapter 10 contains error estimates for Galerkin and collocation solutions of general BIE.

In Chapters 6–9, for each specific type of PDE treated, the physical background is first described, and from there follow the different layer formulations for various types of boundary conditions. Numerical examples and computer graphics are then presented to illustrate the theory.

From our computational experience, our favorite approach is the (augmented) *simple-layer representation collocated by piecewise constant boundary elements*. Among different *admissible* layer representations and discretization schemes, this approach is certainly the easiest to program on a computer. It yields rather smooth profiles of solutions due to the smoothing property of the simple-layer boundary operator. In the discussion of the computational aspects of the examples in Chapters 6–9, we have avoided minute technical details of discretization and quadrature, as readers can easily improvise on their own, also since

such material is already available in existing BEM books (cf. Brebbia's [23, 24], for example). We strongly encourage readers to develop computer packages themselves.

We have tried to present the material as straightforwardly as possible, even though the exposition may occasionally be repetitive and may not be the optimal one. For many beginning engineering graduate students, some sections involving Sobolev spaces and pseudodifferential operators in Chapters 2 and 4 may still appear somewhat technical. Our suggestion for instructors and students alike in such a mathematics/engineering course is to accept the statements of the theorems but to put off the details of the mathematical justification for the first reading. It has been our experience that this will not severely hamper study of the material in Chapters 6–9. Instructors may also assign numerous numerical examples in those chapters as computer projects for students. The computer programs for Chapter 6 are relatively simple. It normally takes about a week to develop a computer program for an example. But the time taken to do this increases with the chapter numbers. Computer programs for the examples in Chapter 9 generally require month-long efforts because of the three-dimensional geometry involved.

We have left out many interesting topics, among them crack and fracture problems, domains with corners, Maxwell's equations in electromagnetism, the linear Stokes flow and time-dependent evolution equations. We hope to be able to treat some of these in a separate volume. Readers may also consult a forthcoming monograph [99] by Hsiao and Wendland for some advanced material on BEM.

The Bibliography contains only those references most pertinent to the content of the book. Based upon the information therein, we hope that interested readers can trace the literature on their own. We apologize in advance for any inadvertent omissions of references, citations and acknowledgements.

Any corrections, comments and suggestions for improvement from readers will be greatly appreciated.

Goong CHEN

Jianxin ZHOU

College Station, Texas June 1991

Dedication

Professor David L. Russell is the former Ph.D. thesis advisor of the first author (G.C.) at the University of Wisconsin-Madison. His work and personality have had a profound influence in shaping both authors' career interests and in developing their professionalism.

Professor George C. Hsiao is a founder of the mathematical theory of boundary element methods. We have benefitted greatly from reading his papers. He also generously consulted with us and provided assistance on numerous occasions.

We have tremendous admiration for both individuals. To them we dedicate this monograph.

Acknowledgements for the 1st edition

Our work in partial differential equations and boundary element methods was supported by National Science Foundation Grants DMS 84-01297 and 87-18510, and Air Force Office of Scientific Research Grants 85-0253, 87-0334, 88-0091 and 91-0097. The Program Managers were Drs John E. Lagnese, Andre Manitius and Michael P. Polis at NSF, and Drs John A. Burns, James M. Crowley and Marc Jacobs at AFOSR. Their interest in our work, their helpful suggestions and generous support are gratefully acknowledged. Dr Andre Manitius was instrumental in arranging a SUN 3/60 Workstation for our research through NSF Grant 87-18510. This equipment proves to benefit us immensely in our computer graphics work.

Supercomputer time was allocated to us by the former John von Neumann Center at Princeton, New Jersey, the National Center for Supercomputing Applications (NCSA) at Urbana-Champaign, Illinois and the Texas A&M University Supercomputer Center. We thank them for their support.

The two authors are extremely fortunate to be able to work closely together at the same institution. In this regard we are particularly indebted to Professors Steven G. Krantz and Jon Pitts, who helped us move to Texas A&M University in 1987.

At various stages of our research and the writing of this book we have benefitted from discussions with many colleagues and collaborators. This list includes Professors Thomas J. Bridges, Michel C. Delfour, Stephen A. Fulling, Nilotpal Ghosh, D. Joe Hartfiel, Wendell H. Mills, E. Bruce Lee, Kaitai Li, Walter Littman, Philip J. Morris, Michael Pedersen, Shunhua Sun and Joseph D. Ward. Professors Andrew Kurdila, Jeff Morgan and Michael Pedersen, Dr Frank Molzahn, Messrs Chih-Hsuan Chen, Yuanhua Deng, Zhonghai Ding and Christopher Peoples have read portions of the manuscript, spotted many errors and offered invaluable suggestions.

The material in § 4.4 was adapted from notes communicated to us from Professor Michael

Pedersen of the Technical University of Denmark, while a large portion of the material in § 5.5 was adapted from Dr Wendell H. Mills' unpublished lecture notes on finite element methods taught at Pennsylvania State University. We are grateful to their contributions.

The computer work on linear elastostatics in Chapter 9 was obtained using the computer packages from Ms Link Ji's Ph.D. thesis directed by the first author at Texas A&M University. Ms Ji has also helped us in many computer-related matters.

Many segments of the material have been taught at Pennsylvania State University and Texas A&M University. During the summer of 1990, by invitation of Professor Sze-Bi Hsu, Director of the Institute of Applied Mathematics, National Tsing Hua University at Hsinchu, Taiwan, Republic of China, the first author gave a short course on boundary element methods. The organization of the book was significantly reshaped after that lecture series. We wish to thank Professor Hsu, all the attendees and students, and the respective institutions for the teaching and lecturing support.

Messrs Kevin Chen and J. Brian Campbell spent many hours in drawing figures and tables. Ms Jean Chen and Robin Campbell patiently typed the long manuscript. We truly admire the excellent quality of their work.

Finally, we deeply thank the Editor, Professor John R. Whiteman, and the editorial staff, Mr Andrew Carrick, Misses Kate Brewin and Fiona Murphy, for all of their efforts and assistance that have made the publication of this volume possible.

Preface to the 2nd edition

Boundary element methods (BEM) continue making the progress since the first edition of this book was published in 1992. The theory and numerical estimates have become more complete, with also an ever increasing number of applications in science and engineering.

Although the advances made in BEM are many-faceted and, in fact, too numerous for us to describe, the authors' main interest in making this updated edition regarding BEM is in adding the theory and applications of BEM to *nonlinear problems*. This is reflected in the new adjusted title of the book: "Boundary Element Methods, with Applications to Nonlinear Problems". Since the last quarter of the 20th Century, beginning during the 1970s, there has been a great shift of interests and emphases in the research and development from the linear to the nonlinear theory, methods and models in all branches of science and engineering. The linear theory of the majority of subjects under study is now widely regarded as reaching an asymptotically complete state of development and has become classical. Nonlinear science and technology are definitely at the frontier of scientific endeavors of the 21st Century. Nonlinear phenomena, such as the lack of uniqueness and stability, the emergence of pattern formations and chaos, etc., now attract the major attention of the scientific community. Therefore, it is imperative for scholars to learn and study the basics of nonlinear science and mathematics. It is based on this conception that, in bringing out the revised edition, we provide two new chapters, Chapters 11 and 12 in this book, to expressly address the theory and applications of BEM to nonlinear elliptic boundary value problems and systems. Therein, we describe the development of various *iterative algorithms* to be coupled with the linear BEM methodology in order to capture solutions of nonlinear equations and systems. Many graphics are accompanied to help visualize the solutions to the nonlinear equations with the hope to aid in the understanding of the physics of those systems.

In making this 2nd edition, we have also tried our best to pick out a number of typographical

errors. A majority of the graphics in the original edition of the book has been refurbished or improved, and we hope they are more eye-pleasing. One section in Chapter 8, the former Section 8.9, has been removed as it is not tightly related to the other parts of the book.

Many of our former collaborators, in particular Dr. Yuanhua Deng, Prof. Zhonghai Ding and Prof. Wei-Ming Ni, have made direct or indirect contributions to the newly added Chapters 11 and 12 in this volume. Ms. Nancy G. Chen and Mr. Changchun Wang have helped reproduce and improve many graphics and photos. Ms. Robin Campbell helped the editing and typesetting of the book. To them, we wish to register our sincerest thanks.

Finally, we express our gratitude to Dr. Keith Jones as Publisher and Zeger as Director at Atlantis Press, for their patience and constant encouragements in making the publication of this volume happen.

Goong CHEN
Jianxin ZHOU

College Station, Texas and Taipei, Taiwan

College Station, Texas

2010

Notation and Abbreviations

$\forall$	for all
$\exists$	there exists
$\equiv$	defined by, or identically equal to
$\perp$	orthogonal to
$ $	absolute value of a real number, or the length of a finite-dimensional vector
$\langle , \rangle$	the inner product of two vectors in a Hilbert space: if $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ are two vectors in $\mathbb{R}^N$, we use either $\langle x, y \rangle$ or $x \cdot y$ to denote $\sum_{j=1}^N x_j y_j$
$\ \ $	the norm of a vector in a Banach space or a Hilbert space
∇	gradient
$\bar{\nabla}$	surface gradient
Δ	Laplacian
$\nabla \cdot$	divergence
Δ^2	biharmonic operator
$\otimes$	tensor product
$\oplus$	direct sum
$\#$	the cardinal number (of a set)
$\hookrightarrow$	continuous injection (imbedding)
dx	the infinitesimal volume element $dx_1 dx_2 \dots dx_N$
$d\sigma_x$	the infinitesimal surface element (with respect to the x variable)
∞	infinity
$*$	for two functions f and g , $f * g$ is the convolution of f with g
a.e.	almost everywhere
BC	boundary condition(s)

BEM	boundary element method(s)
BIE	boundary integral equation(s)
BVP	boundary value problem(s)
c.c.	completely continuous
codim	codimension (of a subspace)
dim	the dimension (of a space)
ess. sup	the essential supremum (of a function)
FDM	finite difference method(s)
FEM	finite element method(s)
iff	if and only if
LHS	the left-hand side (of an equation)
PDE	partial differential equation(s)
RHS	the right hand side (of an equation)
s.t.	such that
w.r.t.	with respect to
arg	the argument of a complex number
$\mathbb{C}$	the set of all complex numbers
$C_0^\infty(\Omega)$	the set of all infinitely differentiable functions with compact support contained in Ω
C^k	k -times continuously differentiable
$C^{k,\alpha}(\overline{\Omega})$	the set of all k -times continuously differentiable functions whose k th derivatives are Hölder-continuous with exponent α on $\overline{\Omega}$
$\mathcal{C}_1, \mathcal{C}_2$	the Calderón projectors
Coker $\mathcal{L}$	the cokernel of a linear operator $\mathcal{L}$
$\mathcal{D}, \mathcal{D}(\mathbb{R}^N)$	the space of test functions ($= C_0^\infty(\mathbb{R}^N)$)
$\mathcal{D}'$	the space of distributions (on $\mathbb{R}^N$)
D^α	the partial differential operator $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}$, for a given multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$, where α_j is a nonnegative integer for $j = 1, 2, \dots, N$
$\frac{\partial}{\partial n}, \partial n, \frac{\partial}{\partial n_x}$	the normal derivative (at x or with respect to the x -variable)
$\delta(x)$	the Dirac delta distribution
$E(x, \xi)$	the fundamental solution (of a partial differential operator)
$\mathcal{F}$	the Fourier transform, $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \langle x, \xi \rangle} f(x) dx$
$\mathcal{F}^{-1}, \overline{\mathcal{F}}$	the inverse (and the conjugate) Fourier transform,

$$\mathcal{F}^{-1}(g)(x) = \overline{\mathcal{F}}(g)(x) = \int_{\mathbb{R}^N} e^{2\pi i \langle x, \xi \rangle} g(\xi) d\xi$$

$\Gamma(x)$	the gamma function
$H_\alpha^{(1)}, H_\alpha^{(2)}$	Hankel functions of the first and the second kinds of order α
$H^s(\Omega)$	the Sobolev space of order s on a domain Ω
$H_{\text{loc}}^s(\mathbb{R}^N)$	the space of all functions or distributions f on $\mathbb{R}^N$ such that $f \in H^s(K)$ for every compact set $K \subset \mathbb{R}^N$
Im	the imaginary part (of a complex number)
Ind	the index (of a Fredholm operator)
$\inf$	infimum
J_α	Bessel function of order α
$\ln$	the natural logarithm function
$L^p(\Omega)$	the space of all p th ($p \geq 1$) power Lebesgue-integrable functions on a domain Ω
$L_{\text{loc}}^p(\mathbb{R}^N)$	the space of all functions whose p^{th} ($p \geq 1$) power is absolutely Lebesgue-integrable on every compact set of $\mathbb{R}^N$
$\mathcal{L}(X, Y)$	the space of all bounded linear operators from space X into space Y
$\max$	the maximum (of a given function or functional): when the supremum is attained (over a given set), we write $\max$ instead of $\sup$
$\text{meas } A$	the measure (or volume) of a given set A
$\min$	the minimum (of a given function or functional): when the infimum is attained (over a given set), we write $\min$ instead of $\inf$
$\mathbb{N}$	the set of all <i>positive</i> integers
$n, n(x), n_x$	the unit normal vector pointing outward on the boundary $\partial\Omega$ of a bounded domain Ω (at $x \in \partial\Omega$)
N_α	Neumann function of order α
$\mathcal{N}(\mathcal{L})$	the null space (or the kernel) of a linear operator $\mathcal{L}$
Ω	an open domain in $\mathbb{R}^N$: Ω is usually simply connected and bounded with smooth boundary $\partial\Omega$; its topological closure is denoted by $\overline{\Omega}$
Ω^c	the complement of $\overline{\Omega}$ in $\mathbb{R}^N$, i.e. $\mathbb{R}^N \setminus \overline{\Omega}$, which is an exterior open domain
$\overline{\Omega^c}$	the closure of Ω^c in $\mathbb{R}^N$
PF	pseudofunction
P.V.	the principal value (of an integral with Cauchy kernel)
$\mathbb{R}$	the set of all real numbers

$\mathbb{R}_+$	the set of all positive real numbers
$\mathbb{R}_-$	the set of all negative real numbers
$\mathbb{R}^N$	the N -dimensional real Euclidean space
$\mathbb{R}_+^N$	the set consisting of all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ where $x_N > 0$
$\mathbb{R}_-^N$	the set consisting of all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ where $x_N < 0$
Re	the real part of a complex number
$\mathcal{R}(\mathcal{L})$	the range of a linear operator $\mathcal{L}$
$S_1 \setminus S_2$	the set consisting of all the elements belonging to set S_1 but not to set S_2
$\mathcal{S}$	the Schwartz space of all rapidly decreasing C^∞ functions (on $\mathbb{R}^N$)
$\mathcal{S}'$	the Schwartz space of distributions
sgn	signature, the sign of a real number
span	the linear span (of a set of vectors)
$\sup$	the supremum (of a given function or functional)
$\text{supp } \phi$	the support of a function or a distribution ϕ
M^{Tr}	the transpose of matrix M
$\text{tr}(M)$	the trace of a matrix M
χ_A	the characteristic function of a set A
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}^+$	the set of all nonnegative integers

Contents

Preface to the 1st edition	vii
Dedication	xi
Acknowledgements for the 1st edition	xiii
Preface to the 2nd edition	xv
Notation and Abbreviations	xvii
1. Introduction	1
1.1 How boundary element methods work	1
1.2 An example of implementation	10
1.3 Comparison between BEM and FEM	14
2. Some Basic Properties of Sobolev Spaces	17
2.1 Definition and imbedding theorems	17
2.2 The trace theorems	29
3. Theory of Distributions	33
3.1 Test functions and generalized functions	33
3.2 The pseudofunctions x_+^{-n} , $n = 1, 2, 3, \dots$	41
3.3 The distributions $(x \pm i0)^{-\lambda}$	44
3.4 Regularizing divergent integrals in $\mathbb{R}^N$	46
3.5 Fourier transform of tempered distributions	49
3.6 Examples of Fourier transforms	53
4. Pseudodifferential Operators	63
4.1 Symbol class S^m	63

4.2	Products and adjoints	67
4.3	Elliptic operators	74
4.4	Calculation of the principal symbols	79
4.5	The Calderòn projector	85
4.6	Fredholm operators	96
4.7	Applications to BIE of elliptic BVP	105
5.	Finite-Element Methods	123
5.1	Minimization of a quadratic functional	123
5.2	Error bounds for internal approximations	128
5.3	Finite-element computation of BVP: an example	135
5.4	(t, m) -systems of approximating subspaces	136
5.5	Polynomial splines in one dimension	137
5.6	Barycentric coordinates	144
5.7	Finite elements in two dimensions	146
5.8	Finite elements in three dimensions	161
5.9	Computation of element matrices	166
5.10	Curved transformations	169
5.11	Accuracy of finite-element approximations	172
5.12	The Aubin–Nitsche lemma	184
5.13	Inverse inequalities	187
6.	The Potential Equation	191
6.1	occurrence of the potential equation	191
6.2	fundamental solution of Laplace equation	194
6.3	volume and boundary potentials	196
6.4	Geometry of hypersurfaces	199
6.5	Regularity of the layer potentials	204
6.6	The two-dimensional case	226
6.7	Regularity solutions of potential BVP	230
6.8	Simple-layer representations	233
6.9	Simple-layer representations for exterior BVP	247
6.10	Double-layer representations for interior BVP	252
6.11	Double-layer representations for exterior BVP	255
6.12	Simple-layer representations for BVP	260

6.13	Double-layer representations for BVP	271
6.14	Multiconnected domains	275
6.15	Direct formulation of BIE	279
6.16	Numerical example (I)	281
6.17	Numerical example (II)	290
6.18	Numerical example (III)	293
7.	The Helmholtz Equation	301
7.1	Background	301
7.2	fundamental solution of Helmholtz equation	308
7.3	Regularity of the layer potentials	312
7.4	Solution of BVP in scattering theory	313
7.5	Asymptotics and uniqueness of solutions	317
7.6	BIE solutions	323
7.7	Exterior impedance BVP	333
7.8	Solutions to the interior BVP	335
7.9	Modified integral equation approach	338
7.10	Numerical example (I)	340
7.11	Numerical example (II)	357
7.12	Numerical example (III)	360
8.	The Thin Plate Equation	373
8.1	Kirchhoff thin static plate model	374
8.2	Existence, uniqueness and regularity	380
8.3	Multilayer potentials for the plate BVP	382
8.4	BIE for interior plate BVP	392
8.5	Other multilayer representations	412
8.6	BIE for exterior plate BVP	414
8.7	Numerical computations and examples (I)	425
8.8	Numerical computations and examples (II)	427
9.	Linear Elastostatics	441
9.1	Derivations equations in linear elasticity	441
9.2	Kelvin's fundamental solution	444
9.3	BVP in linear elastostatics	451

9.4	The Betti–Somigliana formula	457
9.5	Solutions of the interior BVP	462
9.6	BIE in linear elastostatics	468
9.7	Simple-layer representation	472
9.8	Simple-layer solution BVP	478
9.9	Solutions of the exterior BVP	480
9.10	Direct formulations of BIE	487
9.11	Numerical example (I)	492
9.12	Numerical example (II)	499
9.13	Numerical examples (III)	501
10.	Some Error Estimates	507
10.1	error estimates of Galerkin method	508
10.2	degree splines with uniform meshes	517
10.3	Techniques for even- and odd-degree splines	527
10.4	Collocation of augmented systems of BIE	542
11.	BEMs for Semilinear Elliptic PDEs (I)	545
11.1	Introduction	545
11.2	A straightforward iteration scheme	549
11.3	Formulation of boundary integral equations	552
11.4	Galerkin boundary element scheme	559
11.5	Higher than regular-order error estimates	571
11.6	Neumann and Robin boundary conditions	572
11.7	Numerical examples	575
11.8	Quasimonotone coupled 2×2 systems	588
11.8.1	Monotone iteration scheme for nonlinearities of quasimonotone nonincreasing, quasimonotone nondecreasing, and mixed quasi- monotone types	589
11.8.2	Error analysis for a Galerkin boundary element monotone itera- tion scheme	594
11.8.3	Comparison of eigenvalues and ℓ^2 -norms of 2×2 matrices	602
11.8.4	Numerical example	606
12.	BEMs for Semilinear Elliptic PDEs (II)	613

12.1	Introduction	613
12.2	Iterative algorithms and numerical methods	616
12.2.1	The mountain-pass algorithm (MPA)	618
12.2.2	The scaling iterative algorithm (SIA)	622
12.2.3	The direct iteration algorithm (DIA) and the monotone iteration algorithm (MIA)	626
12.2.4	A boundary element numerical elliptic solver based on the simple-layer and volume potentials	627
12.3	Graphics visualization Dirichlet Problem	630
12.3.1	The unit disk	632
12.3.2	Nonconcentric annular domains	633
12.3.3	A “pathological” annulus, with boundary formed by two tangent circles	633
12.3.4	The radially symmetric annulus	637
12.3.5	A dumbbell-shaped domain	644
12.3.6	A starshaped domain degenerated from a dumbbell	647
12.3.7	Dumbbell-shaped domains with cavities lacking symmetry	649
12.3.8	Sign-changing solutions	651
12.4	singularly perturbed Dirichlet problem	657
12.4.1	The unit disk	664
12.4.2	The radially symmetric annulus Ω_6	667
12.4.3	The dumbbell-shaped domain Ω_7	667
12.5	Other Dirichlet problems	672
12.5.1	Henon’s equation	672
12.5.2	Chandrasekhar’s equation	676
12.5.3	The Lane–Emden equation $\Delta u + u^p = 0, p \neq 3$	681
12.6	Sublinear Dirichlet problem	684
12.6.1	Solutions of (12.103) by direct iteration	685
12.6.2	A consequence of visualization: monotonicity of solutions of (12.103) with respect to p	688
Appendix A.		693
A.1	Integration by parts and the Gauss–Green formulas	693
A.2	Banach spaces. Linear operators and linear functionals. Reflexivity	693

A.3	The basic principles of linear analysis	695
A.4	Hilbert spaces. The Riesz representation theorem	696
A.5	Compactness. Completely continuous operators	697
A.6	Quotient spaces	698
A.7	Direct sums. Projection operators	698
A.8	The Cauchy–Schwarz inequality and the Hölder–Young inequality	699
Bibliography		701
Subject Index		711