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Theory of Causal Differential Equations

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Preface

The problems of modern society are both complex and inter-disciplinary. Despite the apparent diversity of problems, however, often tools developed in one context are adaptable to an entirely different situation. For example, consider the well known Lyapunov's second method. This interesting and fruitful technique has gained increasing significance and has given decisive impetus for modern development of stability theory of discrete and dynamic system. It is now recognized that the concept of Lyapunov function and theory of differential inequalities can be utilized to investigate qualitative and quantitative properties of a variety of nonlinear problems. Lyapunov function serves as a vehicle to transform a given complicated system into a simpler comparison system. Therefore, it is enough to study the properties of the simpler system to analyze the properties of the complicated system via an appropriate Lyapunov function and the comparison principle.

It is in this perspective, the present monograph is dedicated to the investigation of the theory of causal differential equations or differential equations with causal operators, which are nonanticipative or abstract Volterra operators. As we shall see in the first chapter, causal differential equations include a variety of dynamic systems and consequently, the theory developed for CDEs (Causal Differential Equations) in general, covers the theory of several dynamic systems in a single framework. Also, many of the same tools which are employed for ODEs (Ordinary Differential Equations) are applicable to CDEs including the method of Lyapunov Functions, for not only the development of stability theory but also other qualitative properties of solutions of CDEs.

It is Volterra who used causal operators implicitly in his work on integral equations and Tonelli gave a sharp definition. It is Tychonoff who made a significant contribution in developing the theory of functional equations involving causal operators. At the end of the last century, the functional equations of a variety of types such as delay differential equations, functional differential equations and integrodifferential equations, etc., were developed and

studied and all these are special cases of causal differential equations .

The first book dealing with functional equations with causal operators is of Corduneanu, which offers some basic theory and paves the way for future development. Recently, a lot of important research has been done in this area and there exists sufficient amount of literature to warrant assembling the existing results in a unified way. It is now widely recognized that this important branch of nonlinear analysis merits further study in order to explore and appreciate the intricacies and advantages involved in the investigation of such dynamic systems.

It is with this spirit we see the importance of the present monograph. As a result, we provide a systematic account of recent developments, describe the present state of the useful theory, show the essential unity achieved and initiate several new extensions to other types of CDEs such as CDEs in infinite dimensional spaces with or without memory, fractional CDEs and set differential equations with causal operators. We hope that this book would motivate scientists to investigate CDEs and their applicability to real world problems and becomes instrumental for further advancement of this interesting area of nonlinear analysis. One can extend CDEs to cover the case of dynamic systems on time scales, impulsive systems and hybrid systems to name a few.

In Chapter 1, we collect preliminary material providing necessary tools, some relevant basic concepts and useful results. Defining causal operators, we give examples of several dynamic systems that are included under the banner of causal systems. Also, we list some useful fixed point theorems, define measures of noncompactness and nonconvexity and indicate necessary preliminaries of a Banach space.

Chapter 2 is devoted to the investigation of fundamental theory of causal differential equations (CDEs) such as comparison principles, existence, uniqueness and continuous dependence of solutions on initial data, existence of extremal solutions, global existence and Nagumo type uniqueness results. Existence of Euler solutions via nonsmooth analysis is also discussed. We present some basic theory for differential and integral equations of Sobolev type and finally inequalities results for causal differential systems.

Chapter 3 contains the method of lower and upper solutions including monotone iterative technique, quasilinearization and their generalizations. Chapter 3, therefore, provides a rich source of methods of finding approximating sequences which are monotone and converge to the extremal solutions/unique solution of the causal differential system.

Chapter 4 introduces the stability theory via Lyapunov method by employing Lyapunov functions, Lyapunov functionals and functions on product spaces. Stability theory in terms

of two measures is indicated and is useful for covering many stability concepts in a single frame work. The method of vector Lyapunov functions and cone valued Lyapunov functions is described to provide weaker sufficient conditions for stability theory.

Finally Chapter 5 deals with several new topics that are initiated in the context of CDEs. They are

- (i) CDEs in a Banach Space,
- (ii) CDEs with fractional derivatives,
- (iii) CDEs with memory,
- (iv) causal set differential equations,
- (v) CDEs with retardation and anticipation.

Few basic results are proved in each case, enough to provide initial apparatus for further study.

Some important features of the monograph are as follows:

- (1) It is the first book that attempts to describe the theory of CDEs as an independent discipline;
- (2) It incorporates the recent general theory of CDEs showing the interconnections between various dynamic systems and CDEs;
- (3) It introduces several new areas of study by providing the initial apparatus for further advancement;
- (4) It is a timely introduction to a subject that is broad enough to cover a variety of dynamic systems in a single setup that follows the present trend of studying analysis and dynamic systems in a general framework.

The monograph will be very useful to those scientists and doctoral students who work in nonlinear analysis in general. It is a good reference book to researchers in several disciplines where real world problems are considered and to graduate students in those disciplines. In fact, the concept of causal operators is prevalent in the engineering literature and thus engineering researchers may find the monograph useful.

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