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The Wavelet Transform

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To my wife
Indrawati Pathak

Preface

The wavelet transform has emerged as one of the most promising function transforms with great potential in applications during the last four decades. The present monograph is an outcome of the recent researches by the author and his co-workers, most of which are not available in a book form. Nevertheless, it also contains the results of many other celebrated workers of the field. The aim of the book is to enrich the theory of the wavelet transform and to provide new directions for further research in theory and applications of the wavelet transform. The book does not contain any sophisticated Mathematics. It is intended for graduate students of Mathematics, Physics and Engineering sciences, as well as interested researchers from other fields.

The Fourier transform has wide applications in Pure and Applied Mathematics, Physics and Engineering sciences; but sometimes one has to make compromise with the results obtained by the Fourier transform with the physical intuitions. The reason is that the Fourier transform does not reflect the evolution over time of the (physical) spectrum and thus it contains no local information. The continuous wavelet transform $(W_\psi f)(b, a)$, involving wavelet ψ , translation parameter b and dilation parameter a , overcomes these drawbacks of the Fourier transform by representing signals (time dependent functions) in the phase space (time/frequency) plane with a local frequency resolution. The Fourier transform is restricted to the domain $L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$, whereas the wavelet transform can be defined for $1 \leq p < \infty$ with appropriate kernel ψ . The frequency resolution is controlled by dilation parameter a , and for small $|a|$, $(W_\psi f)(b, a)$ represents high frequency components of the signal f . Hence, it is desirable to know asymptotic expansions of $(W_\psi f)(b, a)$ for small and large values of the parameters.

The convolution operation associated with an integral transform increases considerably the applicability and theory of the transform. Motivated from the work of I.I. Hirschman

Jr. on Hankel convolution, a theory of wavelet convolution is developed. This provides the interesting property that the product of two wavelet transforms could be a wavelet transform.

The book assumes that the reader has a background in the elements of analysis. Chapter 1 essentially deals with the perquisite material for the theory of distributions and certain integral transforms and related topics. It gives a brief idea about wavelets and wavelet transforms.

In Chapter 2 using properties of Fourier transform certain approximation properties of the wavelet transform are obtained. Certain relations between wavelet transform, Hilbert transform, generalized Hilbert transform and Riesz fractional integrals, and also between their generalizations are obtained. These relations are used to derive new inversion formulae for the wavelet transform of functions belonging to L^p -space under different conditions. Inversion formulae, Parseval formulae, boundedness and approximation results are also obtained in certain weighted L^p -spaces. Some of the results are extended to distributions.

In Chapter 3 it is shown that the wavelet transform is a continuous linear map of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into a similar space $\widetilde{\mathcal{S}}(\mathbb{R}^n \times \mathbb{R}_+)$ when the wavelet belongs to $\mathcal{S}(\mathbb{R}^n)$. The composition of two wavelet transforms is defined and the continuity of the composition operator is investigated. A reconstruction formula for the composition of the wavelet transforms is obtained. Distributional extensions of these transforms are given. Boundedness results regarding these transforms on certain weighted Sobolev spaces are obtained. Abelian theorems are of considerable importance in solving boundary value problems of Mathematical Physics. Abelian theorems for various integral transformations are available in the literature. In this chapter we establish Abelian theorems for the wavelet transform of functions and afterwards derive certain distributional results.

Chapter 4 is devoted to the study of the continuous wavelet transform on certain Gel'fand-Shilov spaces of type S . It is shown that, for wavelets belonging to the one type of S -space defined on $\mathbb{R}$, the wavelet transform is a continuous linear map of the other type of the S -space into a space of the same type (latter type) defined on $\mathbb{R} \times \mathbb{R}_+$. The wavelet transforms of certain ultradifferentiable functions are also investigated.

Chapter 5 contains a study of the continuous wavelet transform on certain Gel'fand-Shilov spaces of type W . The continuity and boundedness results for continuous wavelet transform are obtained on some suitably designed spaces of type W defined on $\mathbb{R} \times \mathbb{R}_+$, $\mathbb{C} \times \mathbb{R}_+$ and $\mathbb{C} \times \mathbb{C}$.

In Chapter 6 the continuous wavelet transform is studied on the generalized Sobolev space $B_{p,k}^\omega$. Boundedness results in this Sobolev space are obtained. Local generalized Sobolev space is defined and some of its important properties are discussed. Wavelet transform with compactly supported wavelet is also studied.

Generalized translation and convolution operators for a general integral transform are defined in order to develop a unified theory of convolutions for all commonly used integral transforms. Thereby, definitions of translation and convolution for continuous wavelet transforms are given and their properties are investigated. A brief account of translation and convolution for discrete wavelet transform is also presented.

A basic function $D(x,y,z)$ associated with general wavelet transform is defined and its properties are investigated. Using $D(x,y,z)$, translation and convolution associated with the wavelet transform are defined and certain existence theorems are proved. An approximation theorem involving wavelet convolution is also proved in Chapter 7.

Chapter 8 contains derivation of a relation between the convolution associated with the wavelet transform and convolution associated with the Fourier transform. This relation is used to define the wavelet convolution transform and study its properties. Existence theorems are proved. We show that the product of two wavelet transforms could be a wavelet transform. Application of the wavelet convolution in approximation of functions is given.

In Chapter 9 asymptotic expansion of the wavelet transform of a function f with respect to the wavelet ψ is derived when the dilation parameter belongs to an open subset of $(0,\infty)$ and the translation parameter goes to infinity. It is assumed that both f and ψ possess power series representations in descending powers of t . Asymptotic expansion of Mexican hat wavelet transform is obtained.

Following Wong's technique asymptotic expansion of the wavelet transform is derived when translation parameter is fixed and dilation parameter goes to infinity and also to zero. Asymptotic expansions for Morlet wavelet transform, Mexican hat wavelet transform and Haar wavelet transform are obtained as special cases in Chapter 10.

The author received inspiration and guidance from Professor C. K. Chui in writing this monograph. His sincere thanks are due to Professor Chui, the Editor of AMES series and Dr. Z. Karssen, Publisher Atlantis Press. The author is also thankful to Professor K. K. Azad (University of Allahabad) for his assistance in so many ways. The author's co-workers G. Pandey, A. Pathak and S. Verma were of great help in the preparation of the

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R. S. Pathak

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