Atlantis Studies in Mathematical Physics: Theory and Applications

Volume 1

Series editor

Norbert Euler, Lulea, Sweden

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Lectures on the Mathematics of Quantum Mechanics I

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ISSN 2211-8055 ISSN 2211-8063 (electronic)
Atlantis Studies in Mathematical Physics: Theory and Applications
ISBN 978-94-6239-117-8 ISBN 978-94-6239-118-5 (eBook)
DOI 10.2991/978-94-6239-118-5

Library of Congress Control Number: 2015937751

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*Whether our attempt stands the test can only
be shown by quantitative calculations of
simple systems*

*Max Born, On Quantum Mechanics
Z. für Physik 26, 379–395 (1924)*

A Caterina, Fiammetta, Simonetta

Contents

Lecture 1: Elements of the History of Quantum Mechanics I	1
1 Introduction	1
2 Birth of Quantum Mechanics. The Early Years	5
3 Birth of Quantum Mechanics 1. The Work of de Broglie	10
4 Birth of Quantum Mechanics 2. Schrödinger's Formalism	12
References	15
 Lecture 2: Elements of the History of Quantum Mechanics II	 17
1 Birth of Quantum Mechanics 3: Born, Heisenberg, Jordan	17
2 Birth of Quantum Mechanics 4. Heisenberg and the Algebra of Matrices	21
3 Birth of Quantum Mechanics 5. Born's Postulate	24
4 Birth of Quantum Mechanics 6. Pauli; Spin, Statistics	25
5 Further Developments: Dirac, Heisenberg, Pauli, Jordan, von Neumann	28
6 Abstract Formulation	29
7 Quantum Field Theory	30
8 Anticommutation Relations	32
9 Algebraic Structures of Hamiltonian and Quantum Mechanics. Pauli's Analysis of the Spectrum of the Hydrogen Atom	33
10 Dirac's Theorem	36
References	37
 Lecture 3: Axioms, States, Observables, Measurement, Difficulties. . . .	 39
1 Introduction	39
2 The Axioms of Quantum Mechanics	40
3 States and Observables	42
4 Schrödinger's Quantum Mechanics	43
5 The Quantization Problem	44
6 Heisenberg's Quantum Mechanics	46
7 On the Equivalence	46

8	The Axioms	47
9	Conceptual Problems	52
10	Information-Theoretical Analysis of Born's Rule	56
	References	58

Lecture 4: Entanglement, Decoherence, Bell's Inequalities,

Alternative Theories	59
1 Decoherence. I	59
2 Decoherence. II	62
3 Experiments	64
4 Bell's Inequalities	66
5 Alternative Theories	69
References	75

Lecture 5: Automorphisms; Quantum Dynamics; Theorems

of Wigner, Kadison, Segal; Continuity and Generators	77
1 Short Summary of Hamiltonian Mechanics	77
2 Quantum Dynamics	79
3 Automorphisms of States and Observables	80
4 Proof of Wigner's Theorem.	82
5 Proof of Kadison's and Segal's Theorems	85
6 Time Evolution, Continuity, Unitary Evolution	87
7 Time Evolution: Structural Analogies with Classical Mechanics.	94
8 Evolution in Quantum Mechanics and Symplectic Transformations	97
9 Relative Merits of Heisenberg and Schrödinger Representations.	99
References	101

Lecture 6: Operators on Hilbert Spaces I; Basic Elements

1 Characterization of the Self-adjoint Operators	107
2 Defect Spaces	110
3 Spectral Theorem, Bounded Case.	112
4 Extension to Normal and Unbounded Self-adjoint Operators	117
5 Stone's Theorem	118
6 Convergence of a Sequence of Operators	120
7 Ruelle's Theorem.	122
References	124

Lecture 7: Quadratic Forms.

1 Relation Between Self-adjoint Operators and Quadratic Forms.	126
2 Quadratic Forms, Semi-qualitative Considerations	127
3 Further Analysis of Quadratic Forms	131
4 The KLMN Theorem; Friedrichs Extension.	133
5 Form Sums of Operators.	137

6	The Case of Dirichlet Forms	138
7	The Case of $-\Delta + \lambda x ^{-\alpha}$, $x \in R^3$	140
8	The Case of a Generic Dimension d	143
9	Quadratic Forms and Extensions of Operators	145
10	A Simple Example.	146
	References	148

Lecture 8: Properties of Free Motion, Anholonomy,

Geometric Phase	151
1 Space-Time Inequalities (Strichartz Inequalities).	153
2 Asymptotic Analysis of the Solution of the Free Schrödinger Equation	154
3 Asymptotic Analysis of the Solution of the Schrödinger Equation with Potential V	156
4 Duhamel Formula	158
5 The Role of the Resolvent.	159
6 Harmonic Oscillator	160
7 Parallel Transport. Geometric Phase	161
8 Anholonomy and Geometric Phase in Quantum Mechanics	163
9 A Two-Dimensional Quantum System	164
10 Formal Analysis of the General Case	165
11 Adiabatic Approximation	166
12 Rigorous Approach	168
References	172

Lecture 9: Elements of C^* -algebras, GNS Representation,

Automorphisms and Dynamical Systems.	173
1 Elements of the Theory of C^* -algebras	173
2 Topologies	177
3 Representations	180
4 The Gel'fand-Neumark-Segal Construction	181
5 von Neumann Algebras.	183
6 von Neumann Density and Double Commutant Theorems. Factors, Weights	184
7 Density Theorems, Spectral Projection, Essential Support	186
8 Automorphisms of a C^* -algebra. C^* -dynamical Systems	188
9 Non-commutative Radon-Nikodim Derivative	192
References	194

Lecture 10: Derivations and Generators. K.M.S. Condition.

Elements of Modular Structure. Standard Form	195
1 Derivations	195
2 Derivations and Groups of Automorphisms	198
3 Analytic Elements	201

4	Two Examples from Quantum Statistical Mechanics and Quantum Field Theory on a Lattice	202
4.1	Example 1	202
4.2	Example 2	205
5	K.M.S. Condition	205
6	Modular Structure	210
7	Standard Cones	212
8	Standard Representation (Standard Form)	213
9	Standard Liouvillian	214
	References	216

Lecture 11: Semigroups and Dissipations. Markov Approximation.

	Quantum Dynamical Semigroups I	217
1	Semigroups on Banach Spaces: Generalities	218
2	Contraction Semigroups	220
3	Markov Approximation in Quantum Mechanics	228
4	Quantum Dynamical Semigroups I	232
5	Dilation of Contraction Semigroups	233
	References	237

Lecture 12: Positivity Preserving Contraction Semigroups

	on C^*-algebras. Conditional Expectations. Complete Dissipations	239
1	Complete Positivity. Dissipations	239
2	Completely Positive Semigroups. Conditional Expectations	243
3	Steinspring Representation. Bures Distance	247
4	Properties of Dissipations	250
5	Complete Dissipations	255
6	General Form of Completely Dissipative Generators	257
	References	259

Lecture 13: Weyl System, Weyl Algebra, Lifting Symplectic

	Maps. Magnetic Weyl Algebra	261
1	Canonical Commutation Relations	261
2	Weyl System	264
3	Weyl Algebra. Moyal Product	265
4	Weyl Quantization	267
5	Construction of the Representations	269
6	Lifting Symplectic Maps. Second Quantization	270
7	The Magnetic Weyl Algebra	274
8	Magnetic Translations in the Magnetic Weyl Algebra	276
	References	281

Lecture 14: A Theorem of Segal. Representations of Bargmann, Segal, Fock. Second Quantization. Other Quantizations

(Deformation, Geometric).	283
1 Fock Space	285
2 Complex Bargmann-Segal Representation	288
3 Berezin-Fock Representation	291
4 Toeplitz Operators	292
5 Landau Hamiltonian Constant Magnetic Field in R^3	293
6 Non-constant Magnetic Field	295
7 Real Bargmann-Segal Representation	297
8 Conditions for Equivalence of Representations Under Linear Maps	299
9 Second Quantization	300
10 The Formalism of Quantization	302
11 Poisson Algebras	302
12 Quantization of a Poisson Algebra	302
13 Deformation Quantization, $*$ -product	304
14 Strict Deformation Quantization	306
15 Berezin-Toeplitz $*$ -product	306
16 “Dequantization”	307
17 Geometric Quantization	309
18 Bohr-Sommerfeld Quantization	311
References	312

Lecture 15: Semiclassical Limit; Coherent States;

Metaplectic Group.	313
1 States Represented by Wave Functions of Class A	315
2 Qualitative Outline of the Proof of (1), (2), (3), (4)	317
3 Tangent Flow, Quadratic Hamiltonians	318
4 Coherent States	319
5 Quadratic Hamiltonians. Metaplectic Algebra.	321
6 Semiclassical Limit Through Coherent States: One-Dimensional Case	322
7 Semiclassical Approximation Theorems	323
8 N Degrees of Freedom. Bogolyubov Operators	327
9 Linear Maps and Metaplectic Group. Maslov Index	330
References	334

Lecture 16: Semiclassical Approximation for Fast Oscillating Phases. Stationary Phase. W.K.B. Method.

Semiclassical Quantization Rules	335
1 Free Schrödinger Equation	335
2 The Non-stationary Phase Theorem	336
3 The Stationary Phase Theorem.	337

4	An Example	340
5	Transport and Hamilton-Jacobi Equations	343
6	The Stationary Case	346
7	Geometric Intepretation.	348
8	Semiclassical Quantization Rules	350
8.1	One Point of Inversion	352
8.2	Two Points of Inversion	353
9	Approximation Through the Resolvent	354
	References	356

Lecture 17: Kato-Rellich Comparison Theorem. Rollnik

	and Stummel Classes. Essential Spectrum.	357
1	Comparison Results	357
2	Rollnik Class Potentials	363
3	Stummel Class Potentials	367
4	Operators with Positivity Preserving Kernels	370
5	Essential Spectrum and Weyl's Comparison Theorems	373
6	Sch'nol Theorem	379
	References	382

Lecture 18: Weyl's Criterium, Hydrogen and Helium Atoms.

1	Weyl's Criterium	383
2	Coulomb-Like Potentials. Spectrum of the Self-adjoint Operator	388
3	The Hydrogen Atom. Group Theoretical Analysis	390
4	Essential Spectrum.	395
5	Pauli Exclusion Principle, Spin and Fermi-Dirac Statistics.	396
5.1	Spin	396
5.2	Statistics	396
5.3	Pauli Exclusion Principle.	397
6	Helium-Like Atoms	398
7	Point Spectrum	401
8	Two-Dimensional Hydrogen Atom.	404
9	One-Dimensional Hydrogen Atom	405
10	Capacity	407
	References	408

Lecture 19: Estimates of the Number of Bound States.

	The Feshbach Method	409
1	Comparison Theorems	409
2	Estimates Depending on Banach Norms	416
3	Estimates for Central Potentials	419
4	Semiclassical Estimates.	420

5	Feshbach Method	424
5.1	The Physical Problem	424
5.2	Abstract Setting	425
	References	428

Lecture 20: Self-adjoint Extensions. Relation with Quadratic Forms. Laplacian on Metric Graphs. Boundary Triples.

	Point Interaction	429
1	Self-adjoint Operators: Criteria and Extensions	429
2	von Neumann Theorem; Krein's Parametrization	432
3	The Case of a Symmetric Operator Bounded Below	436
4	Relation with the Theory of Quadratic Forms	437
5	Special Cases: Periodic, Dirichlet and Neumann Boundary Conditions.	440
6	Self-adjoint Extensions of the Laplacian on a Locally Finite Metric Graph	441
7	Point Interactions on the Real Line.	444
8	Laplacians with Boundary Conditions at Smooth Boundaries in R^3	446
9	The Trace Operator	447
10	Boundary Triples	449
11	Weyl Function.	451
12	Interaction Localized in N Points	452
	References	454
	Index	455

Presentation

These are notes of lectures that I have given through many years at the Department of Mathematics of the University of Rome, La Sapienza, and at the Mathematical Physics Sector of the SISSA in Trieste.

The presentation is whenever possible typical of lectures: introduction of the subject, analysis of the structure through simple examples, precise results in the form of Theorems. I have tried to give a presentation which, while preserving mathematical rigor, insists on the conceptual aspects and on the unity of Quantum Mechanics.

The theory which is presented is Quantum Mechanics as formulated in its essential parts by de Broglie and Schrödinger and by Born, Heisenberg and Jordan with important contributions by Dirac and Pauli.

For editorial reason the volume of Lecture notes is divided into two parts.

The first part, [Lecture 1: Elements of the History of Quantum Mechanics I–Lecture 20: Self-adjoint Extensions. Relation with Quadratic Forms. Laplacian on Metric Graphs. Boundary Triples. Point Interaction](#), contains the essential part of the conceptual and mathematical foundations of the theory and an outline of some of the mathematical techniques that are most useful in the applications. Some parts of these lectures are about topics that are at present subject of active research.

The second volume consists of Lectures 1–17. The Lectures in this second part are devoted to specific topics, often still a subject of advanced research. They are chosen among the ones that I regard as most interesting. Since “interesting” is largely a matter of personal taste other topics may be considered as more significant or more relevant.

At the end of the introduction of both volumes there is a list of books that may be of help for further studies. At the end of each Lecture references are given for self-study.

A remark on the lengths of each Lecture: by and large, each of them is gauged on a two-hour presentation, but the time may vary in relation to the level of preparation of the students. They can also use in self-study, and in this case the amount of time devoted to each Lecture may be vastly different.

I want to express here my thanks to the students who took my courses and to numerous colleagues with whom I have discussed sections of this book for comments, suggestions, and constructive criticism that have much improved the presentation.

In particular, I want to thank Giuseppe Gaeta and Domenico Monaco for the very precious help in editing and for useful comments and Sergio Albeverio, Alessandro Michelangeli, and Andrea Posilicano for suggestions.

Volume I—Basic Elements

Some details of the contents of the Lectures in Volume I:

- *Lectures 1 and 2:* These lectures provide a historical perspective to the beginning of Quantum Mechanics, to its early developments, and to the shaping of present-day formalism.
- *Lecture 3:* An analysis of the mathematical formulation of Quantum Mechanics and of the difficulties one encounters in relating this formalism to the empirical word, mainly for what concerns the theory of measurement.
- *Lecture 4:* Entanglement and the attempts to describe the mathematics of decoherence. An analysis of Bell's inequalities and brief outline of a formalism, originated by de Broglie, in which material points are guided by a velocity field defined by the solution of Schrödinger's equation.
- *Lecture 5:* Groups of transformations of the fundamental quantities in Quantum Mechanics: states and observables. Theorems of Wigner, Kadison, and Segal on implementability with unitary or anti-unitary maps. Continuity of the maps and the basis of Quantum Dynamics.
- *Lecture 6:* Basic facts from the theory of operators in a Hilbert space. Since in Quantum Mechanics these operators represent observables a good control of this formalism is mandatory.
- *Lecture 7:* Elements of the theory of quadratic forms. Quadratic forms are an important tool in the theory of operators on Hilbert spaces and they play a major role in the theory of extensions. Friedrich's extension of a semi-bounded symmetric operator.
- *Lecture 8:* Analytic study of the solutions of the Schrödinger equation, beginning with the simple but instructive case of free motion. Propagation inequalities and their relation to the description of the asymptotic properties of a quantum mechanical system. The problem of anholonomy and the geometric phase.
- *Lecture 9:* Elements of the theory of C^* -algebras and von Neumann algebras. This lecture provides some elements of the theory of automorphisms of C^* -algebras and the description of the dynamics of quantum systems.
- *Lecture 10:* Generators, derivations, and in particular the KMS condition for a group of automorphisms of a C^* -algebra. Implementation of a group of automorphism by a group of unitary operators. Modular structure of a representation and standard form of a von Neumann algebra.

- *Lecture 11*: Basic elements of the theory of semigroups in Banach spaces and of the theory of dissipations. Markov approximation and conditions for its validity. Elements of a converse problem, the *dilation* of a Markov semigroup.
- *Lecture 12*: Role of positivity and complete positivity in the theory of contraction semigroups on C^* -algebras. Elements of the theory of dissipations and basic facts in the theory of Quantum Dynamical semigroups.
- *Lecture 13*: The problem of *quantization*. Weyl system and Weyl algebra, uniqueness theorem of von Neumann and Weyl. Formalism of *second quantization*. Magnetic Weyl algebra.
- *Lecture 14*: Various representations of the Weyl algebra (real and complex representations of Bargmann and Segal, representations of Fock and Berezin). The case of an infinite number of degrees of freedoms. The real representation and the quantization of the free relativistic field (Segal). van Hove's theorem. Brief outline of deformation quantization and of geometric quantization.
- *Lecture 15*: Formally, for systems for which Planck's constant can be considered very small, and for suitable initial data, the dynamics of a system in Quantum Mechanics can be approximated by the dynamics of the corresponding classical system. One refers to this fact by saying that the system can be described in the semiclassical limit. Basic facts regarding this limit. Role of coherent states and metaplectic group.
- *Lecture 16*: Initial data strongly oscillating in configuration space, stationary and non-stationary phase techniques (WKB method). Role of the Maslov index and the origin of the semi-classical quantization rules.
- *Lecture 17*: Deeper analysis of the theory of operators on a Hilbert space, in particular the self-adjoint ones, that describe time evolution. Comparison theorems, in particular the one of Kato-Rellich. Special classes of potentials: the Rollnik and Stummel classes.
- *Lecture 18*: Weyl's criterium for a self-adjoint operator. Detailed study of the Hydrogen and Helium-like atoms, including properties of the spectrum and the presence of embedded eigenvalues.
- *Lecture 19*: An analysis of techniques to estimate the number of bound states for the Schrödinger operator. The Feshbach method is also presented.
- *Lecture 20*: This more specialized lecture develops the theory of self-adjoint extensions of a symmetric operator and its relation to the theory of quadratic forms. Elements of the method of boundary triples, exemplified with the theory of point interaction. Laplacian on a metric graph.

Volume II—Selected topics

Some *tentative* details of the contents of Lectures 1–17

- *Lectures 1 and 2*: Wigner's functions, Pseudo-differential operators. Other quantization procedures (Berezin-Wick, Kohn-Nirenberg, Shubin)
- *Lecture 3*: Shatten class operators, an anthology of inequalities that are commonly used.
- *Lectures 4 and 5*: Mathematics of periodic structures, in particular crystals. Formalism of Bloch-Floquet, Bloch functions, localized Wannier functions. Topological problems connected with the lattice structure.
- *Lectures 6 and 7*: Feynmann-Kac formula. Relation to the heat semigroup, to Wiener's process, and its extension to the Orstein-Uhlenbeck process.
- *Lecture 8*: A brief and elementary treatment of Brownian motion and of Markov processes in general.
- *Lecture 9*: Analysis of the Friedrichs extension of a closed quadratic form. Connection of Dirichlet forms with the theory of self-adjoint operators.
- *Lectures 10 and 11*: Brief outline of the Tomita-Takesaki theory of the Modular Operator in von Neumann algebras and its relation to Friedrichs's extension. Non-commutative integration and non-commutative extension of the equivalence of measures (existence of a Radon-Nykodyn derivative).
- *Lectures 12 and 13*: Elements of scattering theory in Quantum Mechanics. Time-dependent and time-independent formulations. Outline of a method due to V. Enss, which provides a detailed analysis of the evolution in time of the wave function.
- *Lecture 14*: Propagation estimates and Kato's smoothness theory and application to scattering theory. Outlines of Mourre's theory. Further generalizations (e.g., the method of conjugated operators).
- *Lecture 15*: An outline of the theory of a quantum N-body system, both for its spectral properties and the properties of N-body scattering.
- *Lectures 16*: Completely positive maps (open systems). Contractive Dirichlet forms. Markovian and hyper-contracting semigroups, existence, and uniqueness of a ground state. Connection between Markov processes and strictly contractive Dirichlet forms.
- *Lecture 17*: Canonical anticommutation relations. Pauli equation and Dirac operator.

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