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Lectures on the Mathematics of Quantum Mechanics II: Selected Topics

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*Il ne faut pas toujours tellement épuiser
un sujet q'on ne laisse rien à faire au lecteur.
Il ne s'agit pas de faire lire, mais de faire
penser.*

Charles de Secondat, Baron de Montesquieu

A Caterina, Fiammetta, Simonetta

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Introduction to Both Volumes

These books originated in lectures that I have given for many years at the Department of Mathematics of the University of Rome, La Sapienza, and at the Mathematical Physics Sector of the SISSA in Trieste.

I have tried to give a presentation which, while preserving mathematical rigor, insists on the conceptual aspects and on the unity of Quantum Mechanics.

The theory which is presented here is Quantum Mechanics as formulated in its essential parts on one hand by de Broglie and Schrödinger and on the other by Born, Heisenberg, and Jordan with important contributions by Dirac and Pauli.

For editorial reason the book is divided in two parts, with the same main title (to stress the unity of the subject).

The present second volume consists of “[Lecture 1: Wigner Functions. Coherent States. Gabor Transform. Semiclassical Correlation Functions](#)” to “[Lecture 16: Measure \(Gage\) Spaces. Clifford Algebra, C.A.R. Relations. Fermi Field](#)”. Each lecture is devoted to a specific topic, often still a subject of advanced research, chosen among the ones that I regard as most interesting. Since “interesting” is largely a matter of personal taste other topics may be considered as more significant or more relevant.

I want to express here my thanks to the students that took my courses and to numerous colleagues with whom I have discussed sections of this book for comments, suggestions, and constructive criticism that have much improved the presentation.

In particular I want to thank my friends Sergio Albeverio, Giuseppe Gaeta, Alessandro Michelangeli, Andrea Posilicano for support and very useful comments.

I want to thank here G.G. and A.M. also for the help in editing.

Content of Volume II

Lecture 1: Wigner functions. Husimi distribution. Semiclassical limit. WKB states. Coherent states. Gabor transform. Semiclassical limit of joint distribution functions.

Lecture 2: Pseudo-differential operators. Calderon–Vaillantcourt Theorem. $\hbar$ -admissible operators. Berezin, Kohn–Nirenberg, Born–Jordan quantizations.

Lecture 3: Compact, Shatten-class, Carleman operators. Compactness criteria. Radon–Nikodym theorem. Hadamard inequality. Bouquet of inequalities.

Lecture 4: Periodic potentials. Theory of Bloch–Floquet–Zak Wigner–Satz cell. Brillouen zone. Bloch waves. Wannier functions.

Lecture 5: Connection with the properties of a crystal. Born–Oppenheimer approximation. Peierls substitution. The role of topology. Chern number. Index theory. Quantum pumps.

Lecture 6: Lie–Trotter–Kato formula. Wiener process. Stochastic processes. Feynman–Kac Formula.

Lecture 7: Elements of probability theory. Sigma algebras. Chebyshev and Kolmogorov inequalities. Borel–Cantelli lemma. Central limit theorems. Construction of brownian motion. Girsanov formula.

Lecture 8: Diffusions. Ornstein–Uhlenbeck process. Covariance. The infinite-dimensional case. Markov structure. Semigroup property. Paths over function spaces. Markov and Euclidean fields.

Lecture 9: Standard form for algebras and spaces. Cyclic and separating vector. K.M.S. conditions. Tomita–Takesaki theory. Modular operator. Non-commutative Radon–Nikodym derivative.

Lecture 10: Scattering theory. Time-Dependent formalism. Wave operators. Chain rule.

Lecture 11: Scattering Theory. Time dependent formalism. Limit absorption principle. Lippmann–Schwinger equations. The method of Enns. Ruelle’s theorem. Inverse scattering. Reconstruction

Lecture 12: Enns’ propagation estimates. Mourre method. Conjugate operator. Kato smoothness. Double commutator method. Algebraic scattering theory.

Lecture 13: N-Body system. Clusters. Zhislin’s theorem. Spectral structure. Thresholds. Mourre compact operator. double commutator estimates.

Lecture 14: Positivity preserving maps. Ergodicity. Positive improving maps. Contractions. Markov semigroups. Contractive Dirichlet forms.

Lecture 15: Hypercontractivity. Logarithmic Sobolev inequalities. Harmonic group.

Lecture 16: Measure (gauge) spaces. Perturbation theory. Non-commutative integration theory. Clifford algebra. C.A.R. relations. Free Fermi field.

Content of Volume I: Conceptual Structure and Mathematical Background

This first volume consists of Lectures 1 to 20. It contains the essential part of the conceptual and mathematical foundations of the theory and an outline of some of the mathematical instruments that will be most useful in the applications. This introductory part contains also topics that are at present subject of active research.

Lecture 1: Elements of the History of Quantum Mechanics I.

Lecture 2: Elements of the History of Quantum Mechanics II.

Lecture 3: Axioms, States, Observables, Measurement. Difficulties.

Lecture 4: Entanglement, Decoherence, Bell's inequalities, Alternative Theories.

Lecture 5: Automorphisms. Quantum Dynamics. Theorems of Wigner, Kadison, Segal. Generators.

Lecture 6: Operators on Hilbert spaces I: basic elements.

Lecture 7: Quadratic forms.

Lecture 8: Properties of free motion. Anholonomy. Geometric phases.

Lecture 9: Elements of C^* -algebras. G.N.S representation. Automorphisms and Dynamical Systems.

Lecture 10: Derivations and Generators. K.M.S. condition. Elements of modular structure. Standard form.

Lecture 11: Semigroups and dissipations. Markov approximation. Quantum Dynamical semigroups I.

Lecture 12: Positivity preserving contraction semigroups on C^* -algebras. Complete dissipations.

Lecture 13: Weyl system. Weyl algebra. Lifting symplectic maps. Magnetic Weyl algebra.

Lecture 14: Representations of Bargmann-Segal-Fock. Second quantization. Other quantizations.

Lecture 15: Semiclassical limit. Coherent states. Metaplectic group.

Lecture 16: Semiclassical approximation for fast oscillating Phases. W.K.B. method. Semiclassical quantization rules.

Lecture 17: Kato-Rellich comparison theorems. Rollnick and Stummel classes. Essential spectrum.

Lecture 18: Weyl's criterium. Hydrogen and Helium atoms.

Lecture 19: Estimates of the number of bound states. The Feshbach method.

Lecture 20: Self-adjoint extensions. Relation with quadratic forms. Laplacian on metric graphs. Boundary triples. Point interactions.

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