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Volume 4

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Normal and Student's *t* Distributions and Their Applications



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Printed on acid-free paper

To my wife, Masuda

M. Ahsanullah

*To my late parents, Mr. Abdur Rahim
Bhuiyan and Mrs. Ayesha Bhuiyan, and
my wife, Orchi*

B. M. Golam Kibria

*To my late parents, Mr. M. A. Rab and
Mrs. Humaira Khatoon, and my wife,
Nausheen*

M. Shakil

Preface

The normal and Student's t distributions are two of the most important continuous probability distributions, and are widely used in statistics and other fields of sciences. The distributions of the sum, product, and ratio of two independent random variables arise in many fields of research, for example, biology, computer science, control theory, economics, engineering, genetics, hydrology, medicine, number theory, statistics, physics, psychology, reliability, risk management, etc. This has increased the need to explore more statistical results on the sum, product, and ratio of independent random variables. The aim of this book is to study the *Normal and Student's t Distributions and Their Applications*. First, the distributions of the sum, product, and ratio of two independent normal random variables, which play an important role in many areas of research, are presented, and some of the available results are surveyed. The distributions of the sum, product, and ratio of independent Student's t random variables, which are of interest in many areas of statistics, are then discussed. The distributions of the sum, product, and ratio of independent random variables belonging to different families are also of considerable importance and one of the current areas of research interest. This book introduces and develops some new results on the distributions of the sum of the normal and Student's t random variables. Some properties of these distributions are also discussed. A new symmetric distribution has been derived by taking the product of the probability density functions of the normal and Student's t distributions. Some characteristics of the new distributions are presented. Before a particular probability distribution model is applied to fit the real-world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in probability and statistics. We have also provided some characterizations of the family of normal and Student's t distributions.

We hope the findings of the book will be useful for the advanced undergraduate and graduate students, and practitioners in various fields of sciences.

As a preparation to study this book, the readers are assumed to have knowledge of calculus and linear algebra. In addition, they need to have taken first courses in probability and statistical theory.

We wish to express our gratitude to Dr. Chris Tsokos for his valuable suggestions and comments about the manuscript, which certainly improved the quality and presentation of the book. The first author thanks Z. Karssen and K. Jones of

Atlantis Press for the interesting discussions at a meeting in Athens, Greece, for the publication of this book. Summer research grant and sabbatical leave from Rider University enabled the first author to complete his part of the work. Part of the book is from the independent study of the third author with Dr. Kibria. The book was partially written while the second author was on sabbatical in 2010–2011, and he gratefully acknowledges the excellent research facilities of Florida International University. The third author is grateful to Miami Dade College for all the support, including STEM grants. Last but not least, the authors would like to express their deep regret for any error or omission or misprint or mistake, which is very likely to occur in any textbook of this type. We have endeavored our best that our book be typo free (which is impossible but our intention). All suggestions in this regard for improvement in the future are welcome, and will be highly appreciated and gratefully acknowledged.

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