Oscillation Theory for Second Order
Linear, Half-Linear, Superlinear and Sublinear Dynamic
Equations

Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations

by

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Preface

This book is devoted to the qualitative theory of second order dynamic equations. In the last 50 years the *Oscillation Theory* of ordinary, functional, neutral, partial, and impulsive differential equations, and their discrete versions has attracted many researchers. This has resulted in hundreds of research papers in every major mathematical journal. There are many books which deal exclusively with oscillation of solutions of differential equations. However, most of those books appeal only to researchers who already are familiar with the subject. Thus, in an effort to bring oscillation theory to a new and wider audience, in this book we present a compact, thorough, and self-contained account for second order dynamic equations. An important feature of this book is that many examples of current interest are given to illustrate the theory.

In Chapter 1, we introduce oscillation theory of second order differential equations. Here also we provide conditions which guarantee that all solutions of a particular class of second order nonlinear differential equations are continuable, bounded, and converge to zero. Our final section in this chapter states several fixed point theorems which play an important role in establishing existence criteria of nonoscillatory solutions.

It is natural to begin our discussion with second order linear differential equations. The interest in second order linear oscillations is due, in a large part, to the fact that many physical systems are modelled by such equations. We begin Chapter 2 with some of the most basic results in the theory of oscillations of linear ordinary differential equations of second order. In particular, we present Sturm and Sturm–Picone comparison theorems. Then, we provide some necessary and sufficient conditions for nonoscillation as well as some comparison theorems of Sturm type. Also we present sufficiency criteria for nonoscillation of solutions. Next, we establish sufficient conditions for oscillation of second order differential equations with alternating coefficients. Integral averaging techniques and interval criteria are two of the most important concepts in oscillation theory and both are discussed in Chapter 2. Criteria for oscillation of linear second order differential equations with integrable coefficients are also given. We conclude this chapter by discussing the problem of forced oscillations.

In recent years the study of half-linear differential equations has become an important area of research. This is largely due to the fact that half-linear differential equations occur in a variety of real world problems such as in the study of p -Laplace equations, non-Newtonian fluid theory and the turbulent flow of a polytropic gas in a porous medium. In Chapter 3, we present oscillation and nonoscillation criteria for second order half-linear differential equations. We begin with some preliminaries on half-linear differential equations, and then we present Sturm and Levin type comparison theorems and a Liapunov type inequality. Next, we provide an oscillation criterion for almost periodic Sturm-Liouville equations. In this chapter we also present a systematic study of the zeros of solutions of singular half-linear differential equations, nonoscillation characterizations (necessary and sufficient conditions), comparison results and sufficiency criteria for nonoscillation. In addition we study oscillation by employing integral and weighted averaging techniques. Here, interval criteria for the oscillation of half-linear equations will also be provided. Next, we discuss oscillation of half-linear equations with integrable coefficients, and damped and forced equations. We also derive lower bounds for the distance between consecutive zeros of an oscillatory solution. Finally in this chapter, we present a systematic study of the oscillation and nonoscillation of half-linear equations with a deviating argument. Here, classifications of nonoscillatory solutions, and existence results which guarantee that the solutions have prescribed asymptotic behavior will be given.

In Chapter 4, we present a oscillation and nonoscillation theory for second order nonlinear differential equations of superlinear type. We begin with the oscillation of superlinear equations with sign changing coefficients. In particular, we discuss results which involve integrals and weighted integrals of the alternating coefficients, and then provide several criteria which use average behaviors of these integrals. More general averages such as ‘weighted averages’ and ‘iterated averages’ will also be employed. Also, we provide sufficient conditions which guarantee the existence of nonoscillatory solutions, and then we present necessary and sufficient conditions for oscillation of superlinear equations. Oscillation results via comparison of nonlinear equations of the same form as well as with linear ones of the same order will also be presented. We conclude this chapter by discussing oscillation and nonoscillation of forced and damped superlinear differential equations.

In Chapter 5, we present oscillation and nonoscillation criteria for all solutions of second order nonlinear differential equations of sublinear type with alternating coefficients. Our results will involve integrals and weighted integrals of alternating coefficients and in some results we use integral aver-

aging techniques. We also provide some new linearized oscillation results for second order sublinear differential equations. Next, we study the Emden–Fowler equations. These equations originated from the theory of gaseous dynamics in astrophysics around the beginning of the last century, and they also occur in the study of gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics, and in chemically reacting systems. In this chapter we present criteria for nonoscillation of sublinear Emden–Fowler type equations. Also we compare the oscillatory behavior of certain nonlinear equations with related half-linear differential equations. We conclude this chapter by discussing oscillation of general nonlinear differential equations.

In Chapter 6 some special techniques are used to obtain oscillatory criteria for differential equations. Firstly, we present oscillation and nonoscillation theorems for nonlinear second order differential equations by using the method of Olech, Opial and Wazewski. Secondly, we use a variational inequality to study the oscillation of half-linear second order differential equations. Finally, we apply Liapunov second method to provide criteria for oscillation of second order nonlinear equations.

In Chapter 7, we discuss the oscillatory behavior of nonlinear two-dimensional differential systems and second order vector–matrix differential equations. We begin with the oscillation of nonlinear two-dimensional differential systems. This includes the superlinear, linear, and sublinear cases. Then, we discuss oscillation of linear second order differential systems. Two approaches are presented. In the first approach the system considered will be reduced to a certain scalar Riccati inequality, so that known results from the literature can be applied. In the second approach, we will use Riccati and variational techniques. Next, we discuss the oscillation of nonlinear second order differential systems with functionally commutative matrix coefficients. We show that the oscillation theory of such systems can be effectively reduced to the study of diagonal systems of scalar second order differential equations. Then some comparison theorems of Hille–Wintner type for second order operator-valued linear differential equations will be presented. Finally, we discuss oscillation of second order differential systems with a forcing term.

The study of behavioral properties of solutions of differential equations near infinity is of immense importance in applications. In Chapter 8, we present some recent contributions on the asymptotic behavior of solutions of second order differential equations as well as the behavioral properties of positive solutions of singular Emden–Fowler-type equations. It is shown that for a large class of differential equations, not only can the existence of nonoscillatory solutions be proved, but also an explicit asymptotic form of

the nonoscillatory solutions may be provided. Next, by imposing more restrictions on the sign of the integrable coefficient of the equation, we obtain necessary and sufficient conditions so that the solutions have a specified asymptotic behavior as $t \rightarrow \infty$, i.e., solutions which behave asymptotically like a nonzero constant and also those which behave asymptotically like ct , $c \neq 0$. For this, various averaging techniques of the type employed in the previous chapters to study the oscillatory behavior of such equations will be used. Then, we study existence, uniqueness and asymptotic behavior of positive solutions of singular Emden–Fowler–type equations. The cases when the coefficient of the equation under consideration is of constant sign, or of an alternating sign are systematically discussed. Finally, existence and nonexistence results for positive solutions of Emden–Fowler–type systems will be given.

In Chapter 9, some special results are presented. We extend the Sturm–Picone theorem, obtain nonoscillation theorems for perturbed second order nonlinear differential equations, and present a nonlinear Picone type identity which enables us to prove some Sturm–Picone type comparison theorems for nonlinear equations. Next, we study nonoscillatory solutions of forced differential equations of second order. Then, we present some limit cycle criteria and discuss its related properties for nonlinear second order differential equations. Finally, we provide some properties of solutions of very general second order differential equations.

In our final chapter, we introduce a nonoscillation theory for second order differential and neutral inclusions. Here, the results rely on some fixed point theorems for multivalued maps and on a compactness criterion.

We hope this book is timely and complements existing works in the literature on oscillation theory of second order dynamic equations. It can be used as a textbook at the graduate level and hopefully it will stimulate further research in this area.

It is impossible to acknowledge individually colleagues and friends to whom we are indebted for assistance, inspiration and criticism during the preparation of this book. We must, however, express our appreciation and thanks to Sadhna for typing and proof reading parts of the manuscript.

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