# LEARNING MATHEMATICS

*Constructivist and Interactionist Theories of Mathematical Development*

*edited by*

PAUL COBB

*Peabody College, Vanderbilt University,*
*Nashville, Tennessee, U.S.A.*

*Reprinted from Educational Studies in Mathematics 26 (2–3), 1994*

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

Library of Congress Cataloging-in-Publication Data

A C.I.P. Catalogue record for this book is available from the Library of Congress

ISBN 978-90-481-4397-9 ISBN 978-94-017-2057-1 (eBook)
DOI 10.1007/978-94-017-2057-1

*Printed on acid-free paper*

# LEARNING MATHEMATICS: CONSTRUCTIVIST AND INTERACTIONIST THEORIES OF MATHEMATICAL DEVELOPMENT

## TABLE OF CONTENTS

# PREFACE

The first five contributions to this Special Issue on Theories of Mathematical Learning take a cognitive perspective whereas the sixth, that by Voigt, takes an interactionist perspective. The common theme that links the six articles is the focus on students' inferred experiences as the starting point in the theory-building process. This emphasis on the meanings that objects and events have for students within their experiential realities can be contrasted with approaches in which the goal is to specify cognitive behaviors that yield an input-output match with observed behavior. It is important to note that the term 'experience' as it is used in these articles is not restricted to physical or sensory-motor experience. A perusal of the first five articles indicates that it includes reflective experiences that involve reviewing prior activity and anticipating the results of potential activity. In addition, by emphasizing interaction and communication, Voigt's contribution reminds us that personal experiences do not arise in a vacuum but instead have a social aspect.

In taking a cognitive perspective, the first five contributions analyze the processes by which students conceptually reorganize their experiential realities and thus construct increasingly sophisticated mathematical ways of knowing. The conceptual constructions addressed by these theorists, ranging as they do from fractions to the Fundamental Theorem of Calculus, indicate that experiential approaches to mathematical cognition are viable at all levels of mathematical development. Although the authors use different theoretical constructs, several additional commonalities can be discerned in their work. For example, all would seem to concur with Steffe and Wiegel's claim that the learning environments that students establish are a function of the concepts and operations they use when interpreting situations. Further, all are concerned with what is colloquially called meaningful learning – learning that involves the construction of experientially-real mathematical objects. In addition, all trace the origin of these mathematical objects to students' activity, both physical and conceptual. Thus, Sfard and Linchevski discuss the process of reification, and Steffe and Wiegel analyze students' internalization and interiorization of their activity. In taking this approach, the authors characterize knowledge as action – they are concerned with students' mathematical ways of knowing. Pirie and Kieren, for example, stress that mathematical understanding is a process, not an acquisition or a location. Similarly, Confrey and Smith differentiate between the various types of additive and multiplicative units that students construct in terms of the actions that generate them. Thompson,

*Educational Studies in Mathematics* **26:** 105–109, 1994.

for his part, stresses that his focus is on the conceptual operations and imagery that constitute students' understanding of particular mathematical topics.

A further commonality that cuts across the contributions concerns the importance attributed to imagery. For example, Steffe and Wiegel talk of students representing their mathematical activity by running through it in thought. Confrey and Smith refer to an experiential dimension wherein children distinguish more, less, and the same rates of change non-numerically, perhaps when sensing a change of speed while riding in an automobile. In Confrey and Smith's view, imagined sensations of this type constitute one of the bases from which more sophisticated conceptions of rates of change develop. Consistent with the general emphasis on knowing rather than knowledge, both Thompson and Pirie and Kieren argue that images are dynamic aspects of the understanding process rather than static mental pictures. Both also stress that the role imagery plays in mathematical activity evolves as particular concepts become increasingly abstract. Thus, Thompson draws on Piaget's work to distinguish between three types of images. The third of these types of images appears to be reflexively related to the student's mental operations in that the image is shaped by the operations, and the operations are constrained by the image. Similarly, in discussing a process termed folding back that occurs when the student is faced with a problem or question that is not immediately solvable, Pirie and Kieren argue that the resulting image is not the same as images formed earlier in development because it is informed by more abstract interests and understandings. It should be noted that the general focus on imagery is not restricted to elementary mathematical concepts. Thompson in fact locates students' difficulties with the Fundamental Theorem of Calculus in their impoverished images of rate, and Pirie and Kieren argue that more advanced mathematics needs to be worked out at the level of image making before students can begin to look for formalization or structure.

An additional point of contact between the authors concerns the pragmatic approach they take to theorizing. Thus, Thompson argues that a theory is an anticipatory scheme that enables the researcher to imagine problems that might arise in the learning-teaching situation, and to plan potential solutions to them. In a similar vein, Steffe and Wiegel say that their overall objective is to develop what they call the technical knowledge necessary for teaching. This theoretical pragmatism is reflected in the methodologies used by the researchers. All involve the intensive analysis of students' mathematical activity, resulting in theoretical constructs that are empirically grounded in a non-empiricist sense.

Both Thompson's and Sfard and Linchevski's contributions indicate that historical analyses complement experiential approaches to cognition. Thompson's motivation for analyzing Newton's and Leibniz's work is to look for kinds of reasoning that might provide a starting point for instruction oriented to the development of imagery and forms of expression that could support later insights into central ideas of the calculus. Sfard and Linchevski's objective in analyzing the historical development of algebra is to identify possible developmental stages that both clarify students' difficulties and inform instruction. In light of this demonstrated relevance of historical analyses, Sfard and Linchevski are careful to note that caution should be exercised when drawing analogies between historical and psychological development because a deliberately guided process of reconstruction might not necessarily follow the meandering path of the first travelers.

A further emerging theme apparent in the contributions concerns the role that language and symbols play in mathematical development. Sfard and Linchevski's historical analysis of algebra indicates that the introduction of symbols is a central aspect of the reification process by which mathematical activity is objectified. While acknowledging that the development of symbols is by itself insufficient, they nonetheless contend that symbols are manipulable in a way that words are not, and this makes it possible for algebraic concepts to have an object-like quality. Pirie and Kieren make a related argument when they claim that both acting and expressing are necessary at any level of understanding before a student can move on to another level. Their observations of students indicate that in the absence of an expression of understanding, mathematical notions based solely on acting are ephemeral and do not remain with students from one session to the next. This leads Pirie and Kieren to conclude that acting and expressing constitute a complementarity at each of the levels of understanding they have identified. Confrey and Smith's analysis of the various types of units that students construct while solving problems that involve rates of change succintly illustrates this complementarity. They note that unit names from the additive counting world are not relevant to a multiplicative splitting world such as that in which a student shares a pie by splitting it in half, and then splits the halves in half, and so on. As part of their analysis, they develop names for the process of generating units in this multiplicative world. For example, they describe the process by which a student cuts a pie into eight equal pieces as three multiplicative units of two, or three 2-splits.

This discussion of symbolizing and expressing hints at the social aspect of mathematical development. Thus, Pirie and Kieren note that students express their understanding for others as well as themselves. More generally, ways of symbolizing and expressing are negotiated in the course of social interactions. It is in regard that Voigt's presentation of his interactionist perspective complements the cognitive perspective taken in the first five papers. Voigt's basic claim is that the social dimension is intrinsic to mathematical development because the teaching-learning process is an interpersonal process. Both Steffe and Wiegel, and Pirie and Kieren touch on this social dimension when they discuss the influence of the teacher's interventions on students' goals and interpretations. Voigt views these interventions as actions in an ongoing process of negotiation and argues that mathematical meanings are necessarily matters for negotiation because every object and event is potentially ambiguous and plurisemantic. Voigt also contends that negotiation occurs even when the teacher and students do not explicitly argue from different points of view. Further, the sample episodes he presents indicate that it is not only the students who develop novel understandings in the course of these interactions, but the teacher also modifies his or her interpretations while negotiating meanings. This point is made particularly forcefully by Confrey and

Smith when they note that they, as teachers, reformulated their own mathematical understanding as they interpreted students' approaches and methods in the course of a teaching experiment.

Several of Voigt's examples deal with the process of mathematization by which an empirical situation is transformed into a mathematical one. Steffe and Wiegel also address this issue in some detail when they discuss how a student's cognitive play is transformed into mathematical activity. A comparison of these analyses highlights the complementary nature of the cognitive constructivist and interactionist perspectives. Steffe and Wiegel's focus is on the ways in which the researcher influences the student's interpretations during a teaching experiment. Their analysis is made from the perspective of someone who is inside the interaction and is concerned with the ways in which the student modifies his or her own activity in the course of the interaction. In contrast, Voigt's analysis is made from the outside, from the point of view of someone who is an observer of rather than a participant in ongoing interactions. From this perspective, the focus is on the taken-to-be-shared meanings that emerge between the teacher and students rather than on the meanings of any individual participant. These taken-to-be-shared meanings can be thought of as constituting an evolving consensual domain for mathematical communication. As Voigt makes clear, taken-to-be-shared meanings are not cognitive elements that capture, say, the partial match of individual interpretations but are instead located at the level of interaction. Thus, whereas for Steffe and Wiegel, mathematization is a process of individual conceptual reorganization, for Voigt it is a process of negotiation in the course of which the teacher and students collectively modify what is taken-to-be-shared between them. The complementarity between the two positions becomes apparent when it is noted that taken-to-be-shared or consensual meanings are established by the teacher and students as they attempt to coordinate their individual activities. Conversely, the teacher's and students' participation in the establishment of taken-to-be-shared mathematical meanings both supports and constrains their individual interpretations.

Both Sfard and Linchevski's and Thompson's contributions draw attention to a further feature of the social aspect of mathematical development. Sfard and Linchevski report that the students in their study frequently evidenced semantically-debased conceptions in which algebraic formulae were viewed as nothing more than strings of symbols to which certain procedures are routinely applied. Similarly, Thompson observed that the supposedly mathematically-sophisticated students who participated in this teaching experiment tended to use notation opaquely by simply associating patterns of routine figural actions with various notational configurations. Thus, as Thompson notes, although his primary focus was on students' evolving cognitions, he found himself negotiating what it means to know and do mathematics with them. In this regard, Voigt stresses that students' learning is not restricted to mathematics. They also learn how to negotiate mathematical meanings with the teacher and, in the process, develop beliefs about what counts as a problem, a solution, an explanation, and a justification. Consequently, from Voigt's interactionist perspective, Thompson can be seen to be initiating and guiding the renegotiation of the norms that constitute the classroom mathematical microculture. This process of renegotiation appears to have been crucial given Thompson's interest in meaningful mathematical activity that involves attempts to interpret notation by establishing imagery. Thompson's experiences indicate both the value of long-term teaching experiments and the need to consider the social dimensions of mathematical development even when the research focus is primarily cognitive.

One final issue that cuts across all six articles is that of attending to what Confrey and Smith call the student's voice. Relatedly, Steffe and Wiegel argue that students will not sustain mathematical activity unless they experience satisfaction in the course of that activity. The transcripts presented in the five cognitive papers indicate that, for the most part, the students did experience satisfaction as they engaged in mathematical activity with the researchers. Voigt, speaking from the interactionist perspective, observes that students are necessarily active participants who make original contributions to the establishment of consensual meanings. As the cognitive papers indicate, instructional approaches compatible with constructivism attempt to bring this originality to the fore by viewing it as a resource to be capitalized on rather than as an impediment to be pushed underground.

*Peabody College*
*Vanderbilt University*
*Nashville, TN 37203*
*U.S.A.*

PAUL COBB