Analytic $\mathcal{D}$-Modules and Applications

Mathematics and Its Applications

Volume 247

Analytic $\mathcal{D}$-Modules and Applications

by

Jan-Erik Björk
Department of Mathematics,
Stockholm University,
Stockholm, Sweden

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

Library of Congress Cataloging-in-Publication Data

Björk, Jan-Erik.
Analytic D-modules and applications / by Jan-Erik Björk.
p. cm. -- (Mathematics and its applications ; v. 247)
Includes index.
ISBN 978-90-481-4238-5 ISBN 978-94-017-0717-6 (eBook)
DOI 10.1007/978-94-017-0717-6
1. Differentiable dynamical systems. 2. D-modules. 3. Complex manifolds. 4. Differential topology. I. Title. II. Series: Mathematics and its applications (Kluwer Academic Publishers) ; v. 247.
QA614.8.B58 1993
514'.74--dc20 92-44826

ISBN 978-90-481-4238-5

Typeset by AMS-TEX

Printed on acid-free paper

SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'
Jules Verne

The series is divergent; therefore we may be able to do something with it.
O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded nonsense'.
Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and nonlinearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

Analytic D-module theory studies holomorphic differential systems of equations on complex manifolds. It is a quite new area in mathematics that has drawn much attention in recent years and that has, correspondingly, enjoyed vigorous development. Together with its tightly interlined algebraic counterpart it has brought new insights and methods in many areas such as representation theory, hypergeometric functions, intersection cohomology, and residues for several complex variables, and it has been instrumental in the solution of some outstanding conjectures.

The present book is the first monograph on analytic D-modules and it offers a complete and systematic treatment of the foundations together with a thorough discussion of such modern topics as the Riemann-Hilbert correspondence, Bernstein-Sata polynomials and a large variety of results concerning micro-differential analysis.

The book has been long in the making. Preliminary versions in various stages of completeness have circulated for many years among the specialists, and many improvements resulted from their reactions and comments. It is a real pleasure to welcome the finished product, impressive as it is.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Amsterdam, October 1992

Michiel Hazewinkel

Table of contents

Preface

Analytic $\mathcal{D}$-module theory treats holomorphic differential systems on complex manifolds, presented by bounded complexes of sheaves of modules over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators on a given complex manifold X. The theory has a broad range of applications to other areas than PDE-theory in the complex domain. Most notable is the use in representation theory, particularly infinite dimensional representations of semisimple complex or real Lie groups. Another aspect is the inverse process where one starts with objects such as some family of generalized hypergeometric functions and asks for the class of differential operators for which the given family yield homogeneous solutions. This leads naturally to sheaves of $\mathcal{D}$-modules. Here is also the Riemann-Hilbert correspondence which relates $\mathcal{D}$-module theory with constructible sheaves and algebraic topology on analytic spaces.

$\mathcal{D}$-module theory has mainly been developed since 1970. The initial contributions were done at RIMS in Kyoto based upon pioneering ideas of M. Sato. The present book offers a synthesis of research articles during the last two decades and contains detailed proofs of all foundational results in analytic $\mathcal{D}$-module theory.

The study of regular holonomic systems is a central topic in this book. Chapter 7 contains an inverse to the de Rham functor in the Riemann-Hilbert correspondence which gives important insight into the class of distributions on the underlying real manifold $X_{\mathbf{R}}$ which are solutions to homogeneous regular holonomic systems. This has for example applications to calculus of residues of several complex variables. The last chapter contains a study of micro-differential systems and a proof of the deep fact that the microlocalisation of a regular holonomic $\mathcal{D}$-module remains regular holonomic.

Obvious lack of space prevents an extensive study of applications and examples to the general theory. Instead the overall ambition has been to provide a reasonably self-contained presentation aimed at the non-expert which can serve as the necessary background for a future studies of special topics in the subject.

To consolidate this an extensive appendix has been included devoted to the most important tools which are used in $\mathcal{D}$-module theory. It contains an account of sheaf theory in the context of derived categories, a detailed study of filtered non-commutative rings and homological algebra, the basic material on symplectic geometry and stratifications on complex analytic sets.