REASONING IN QUANTUM THEORY

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Sharp and Unsharp Quantum Logics

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Preface

The term "quantum logic" has entered our languages as a synonym for something that doesn't make sense to our everyday rationality. Or, somewhat more technically but still in the common literature, it signifies some generic sort of mystification of classical logic understood only by the illuminati. In the technical literature, it is most frequently used to designate the set of projections $\Pi(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ or the set of positive operators $\mathcal{E}(\mathcal{H})$ which fall between the smallest and the largest projections on $\mathcal{H}$ in a suitable ordering - or some algebraic generalization of one of these.

Thus, we have two concrete or standard quantum logics. These are structures closely related to the usual mathematical formalism that underlies the foundation of quantum theory (QT). The set $\Pi(\mathcal{H})$ is the basis for the sharp theory and $\mathcal{E}(\mathcal{H})$ for the unsharp theory, in much the same way that classical logic is based, in its sharp and unsharp manifestations, on (subalgebras of products of) the two-element set $\{0,1\}$ and the real unit interval $[0,1]$, respectively.

There is a still unfolding panorama of structures that generalize these two standard models. The theory of the foundations of quantum mechanics called "quantum logic" studies the standard models and their abstractions.

Confusion has persisted as to just what quantum logic is and how it should be construed as a veritable logic. The purpose of this book is to delineate (what we know of) the quantum logics, to explain of what the panorama of quantum logics consists and to present actual logics whose algebraic or Kripkean semantics are based on the algebraic models that are historically referred to as quantum logics. Our position is that there is not one but that there are many quantum logics. These logics have various models, usually one of the orthomodular structures, which include orthomodular lattices, orthomodular posets, orthoalgebras, and effect algebras.

We present sufficient historical background to give the reader an idea of how the theory developed. However, far more is presented than is needed to simply develop the logic, so that the novice may pick up the motivating aspects of the subject. Readers not interested in the technical logical details of quantum logics may gain an accounting of the mathematical aspects of "quantum logic," the models of the theory and how they relate to one another, by reading only the initial chapters. Readers wanting to learn more about these algebraic structures are referred to (Kalmbach, 1983; Dvurečenskij and Pulmannová, 2000).

Even the purist (non-quantum) logician may have some difficulty reading the technical logic in the second half of the book without the earlier preparation; such readers could, however, begin in the logical sections and refer to the earlier sections as needed. Quantum logicians, however, may proceed directly to the latter chapters.

We are writing for a multidisciplinary audience, and we warn the reader that we at times are too verbose for mathematicians, too pedantic for physicists, too glib for logicians, and too technical for philosophers of science. We assure the reader that we have had the whole readership in mind as we made our compromises and we beg her indulgence.

Here is an outline of the organization of the book. There are two parts. Part I, which consists of the first seven chapters, presents the historical background and the algebraic developments that motivate the syntax and underlie the semantics of the sequel. Part II studies a variety of quantum logics; in this Part, the term "logic" is used in the traditional sense, as a theory for a consequence relation that may hold between well-formed formulas of a given language. Semantical characterizations are introduced, both algebraic semantics and Kripkean semantics.

We set the stage in Chapter 1 by presenting some abstract notions needed later and by sketching the historical underpinnings of the subject. We present some of the basic notions of ordered sets followed by a quick review of Hilbert space and operators thereon. We recall von Neumann's axioms for quantum theory and Birkhoff and von Neumann's inseminal idea that propositions about quantum systems can be viewed as forming a kind of logic more appropriately modeled by projections on a Hilbert space than by a Boolean algebra. Then, we present von Neumann's axiomatic scheme which is followed by a discussion of Mackey's treatment and the abstractions that derived from it.

Chapter 2 presents the abstract axiomatization of sharp quantum theory, as well as the notion of *event*. The events, it is argued, band together to form a σ-complete orthomodular poset. The abstract notion of state and observable are given, and the previously developed structure is rephrased in an equivalent way as an *event-state system*. The interplay between events and states is further developed in *preclusivity spaces* and *similarity spaces*. A preclusivity space is a graph $(X, \perp)$, where the relation $\perp$ is symmetric and irreflexive; the corresponding similarity space is the pair $(X, \mathcal{R})$, where $\mathcal{R} = X \times X - \perp$. Preclusivity spaces invariably induce a closure operator $Y \mapsto Y^{\perp\perp}$, for $Y \subseteq X$, and the family of all closed sets $\{Y : Y = Y^{\perp\perp}\}$ forms a complete ortholattice.

The notion of the **Yes**-set, **Yes**(E), associated with each event E is introduced. **Yes**(E) is the set of all states s with $s(E) = 1$. By regarding states as *possible worlds*, this notion later allows us to introduce a Kripkean semantics.

The main purpose of Chapter 3 is to introduce and characterize the complete ortholattice $\Pi(\mathcal{H})$ of all projections of a Hilbert space. We are led to view $\Pi(\mathcal{H})$ and the states $\mathcal{S}$ thereon as an event-state system, appropriately baptized *the Hilbert event-state system.* The basic lattice theoretic properties of $\Pi(\mathcal{H})$ are given, and we sketch the development of Piron, Keller, Solèr, Holland, and Morash which some regard as the crowning mathematical achievement of the quantum logic approach to the foundations of quantum mechanics, namely the characterization of $\Pi(\mathcal{H})$ by lattice theoretic properties.

In Chapters 4 and 5, we introduce $\mathcal{E}(\mathcal{H})$, the (possibly) unsharp extension of $\Pi(\mathcal{H})$. The underlying set of $\mathcal{E}(\mathcal{H})$ is the set of all the bounded operators trapped between $\mathbb{O}$ and $\mathbb{1}$ in the usual ordering of self-adjoint operators. It forms an *effect algebra*, a type of structure that is then studied in some detail, and it is called *the standard unsharp effect algebra.* Along the way, we introduce *BZ-posets* and *effect BZ-posets*, the Mac Neille completion, and unsharp preclusivity spaces. The BZ-effect algebras admit 2 complements, a Brouwer complement and the fuzzy complement of the effect algebra. Having both allows us to define the necessity and possibility operators.

The Łukasiewicz operations of disjunction and conjunction are introduced and discussed. Axioms for MV algebras are given. These provide an adequate semantic characterization for Łukasiewicz' many valued logics. The quantum version follows, which consists of the axioms for what are called QMV algebras. The QMV algebras that correspond to effect algebras are determined; they are called *quasi-linear QMV algebras.*

Chapter 6 presents the corresponding abstract axiomatics for unsharp QT. Chapter 7 investigates alternative notions for sharpness and the connections between them.

We begin Part II with a discussion of the nature of algebraic semantics and Kripkean semantics. We give the algebraic characterization and the Kripkean characterization for classical logic (**CL**) as well as for intuitionistic logic (**IL**), the former being provided by the Boolean algebras and the latter by Heyting algebras. We review the notions of truth, logical truth, consequence, and logical consequence in both **CL** and **IL**; and we recall that the algebraic semantics and the Kripkean semantics of **CL** characterize the same logic, i.e., the notion of logical truth (resp., logical consequence), for the algebraic semantics is equivalent to that of the Kripkean semantics; the same is true for **IL**.

In Chapter 8, we proceed to sharp quantum logic (**QL**). There are two: **OL** and **OQL**. In **OL** (resp., **OQL**), the algebraic semantics are classified by the class of all algebraic realizations based on ortholattices (resp., orthomodular lattices). The similarity spaces discussed earlier provide the basis for the Kripkean semantics of **OL**. Although the details of the constructions are not the same as for **CL** and **IL**, the algebraic semantics for **OL** (resp., **OQL**) and the Kripkean semantics of **OL** (resp., **OQL**) characterize the same logic. We then specify **OQL** to the Hilbert event-state systems to obtain the algebraic and Kripkean realizations for a quantum system $\mathfrak{S}$ with associated Hilbert space $\mathcal{H}$.

We then turn to a discussion of polynomial implications in **OQL**, arguing that there is a favored one, the so-called Sasaki hook. This conditional may be interpreted as a counterfactual conditional.

In Chapter 9, we introduce the notion of a quasi-model in both the algebraic and the Kripkean semantics of **QL**. This notion allows us to discuss three interesting metalogical properties, called Herbrand-Tarski, Verifiability, and Lindenbaum, whose failure in **QL** represents a significant anomalous feature of quantum logics.

Chapter 10 addresses syntax. In it we provide an axiomatization of **QL** in the natural deduction style, proving soundness and completeness theorems with respect to the Kripkean semantics presented earlier. **OQL**, but not **OL**, admits a material conditional, and we prove a deduction theorem for **OQL**.

In Chapter 11, we prove that orthomodularity is not a first-order property. We also introduce *Hilbert quantum logic* (**HQL**) which is the logic that is semantically based on the class of all Hilbert lattices. We argue that **HQL** is strictly stronger than **OQL**.

In Chapter 12, we extend *sentential quantum logic* presenting an algebraic realization and a Kripkean realization for (first-order) **QL** with its attendant notions of satisfaction, verification, truth, logical truth, consequence in a realization, logical consequence, and rules of inference. We briefly discuss *quantum set theory* in which the usual Boolean-valued models are replaced by complete orthomodular lattices, perhaps the most striking feature of which is the failure of the Leibniz-substitutivity principle. Other approaches to quantum set theory are discussed, where the notion of extensional equality is not characterized by membership. Not only the extensionality axiom but Leibniz's principle of indiscernibles may be violated.

Proceeding from the total logics, those which are syntactically closed under the logical connectives and in which any sentence always has a meaning in both the algebraic and the Kripkean semantics, we present in Chapter 13 *partial classical logic* (**PaCL**) in which sentences need not have a meaning. An interesting feature of **PaCL** is the following: some characteristic classical laws (like distributivity) that are violated in standard quantum logic represent logical truths of **PaCL**.

After presenting an axiomatization of **PaCL** and proving soundness and completeness theorems, we discuss the failure of the Lindenbaum property. We introduce a relativization (to some nonempty class of realizations) of this notion and relate this relativized notion to the relative validity of the logical truths of classical logic as well as to the question of (non-contextual) hidden variable theory.

In Chapter 14, we turn our attention to unsharp quantum logics. In sharp logics both the logical and the semantic version of the noncontradiction principle hold. This means that $\alpha \curlywedge \neg\alpha$ is always false (in **PaCL**, when defined), and it never happens that both α and $\neg\alpha$ are true. In unsharp logics both conditions fail; thus, these logics are the natural logical abstractions of the effect-state systems studied in Part I.

Paraconsistent quantum logic (**PQL**) is presented first. This logic has an algebraic semantics based on bounded involution lattices and a Kripkean semantics in which the accessibility relation is symmetric but not necessarily reflexive, while the propositions behave as in the **OL** case. The other semantic definitions agree with **OL**, *mutatis mutandis*, and the algebraic and Kripkean semantics characterize the same logic. The axiomatization of **PQL** looks like that of the **OL** calculus except that the absurdity rule and the Duns Scotus rule are lacking. As in **OL**, the logic **PQL** satisfies the finite model property and is therefore decidable.

Regular bounded involution lattices give rise to a specialization of **PQL**, naturally called *regular paraconsistent quantum logic* (**RPQL**). The Kripkean realizations of **RPQL** may be represented by ε-preclusivity spaces in which the preclusivity relation is sensitive only to a characteristic approximation degree ε. The realizations are sharp when $0 \leq \varepsilon < \frac{1}{2}$ and unsharp when $\frac{1}{2} \leq \varepsilon \leq 1$.

In Chapter 15, we turn to a stronger unsharp quantum logic, the *Brouwer Zadeh logics*. These are natural abstractions of the class of all BZ-lattices studied in Chapter 4.

In Chapter 16, we study *partial quantum logics* (**PaQL**). There are three types: *unsharp* **(UPaQL)**, *weak* **(WPaQL)** and *strong* **(SPaQL)**. They admit a semantic characterization corresponding to the class of all effect algebras, orthoalgebras and orthomodular posets, respectively.

The language of **PaQL** consists of a set of atomic sentences and of two primitive connectives, the *negation* $\neg$ and the *exclusive disjunction* $\forall$ (aut), while the *conjunction* is metalinguistically defined, via the de Morgan law. Whereas all disjunctions and conjunctions are considered "legitimate" from a linguistic point of view, a disjunction $\alpha \forall \beta$ will have the intended meaning only in case the values of α and β are orthogonal in the corresponding effect algebra, and when this is not the case an arbitrary meaning is assigned.

First, we give the semantics of **UPaQL**, then the axiomatization which, unlike **QL**, admits only inferences from single sentences. Then, we add the Duns Scotus rule to get an axiomatization for **WPaQL**. Finally, we give an axiom which essentially states that the disjunction $\alpha \forall \beta$ behaves like a supremum whenever α is orthogonal to β, arriving at **SPaQL**. We then discuss soundness and completeness theorems.

We finish the chapter with a discussion of the *Łukasiewicz quantum logic* (**ŁQL**), which generalizes both **OQL** and $\mathbf{L}_{\aleph}$ (Łukasiewicz' infinite many valued logic). The semantics of **ŁQL** is based on QMV algebras.

Finally, in Chapter 17, we discuss *quantum computational logic*. The quantum generalization of a bit is called a quantum bit, or qubit; it is a unit vector in the Hilbert space $\mathbb{C}^2$. Quantum computations, or quantum logical gates, are represented by unitary transformations on qubit systems. Fixing an orthonormal basis, called *the computational basis*, we define an *n-qubit system* to be an element of the n-fold tensor product $\otimes^n\mathbb{C}^2$. Quregisters are qubits or n-qubit systems.

The quantum computational logic is based on certain quantum logical gates defined on quregisters, the Toffoli gate, a reversible conjunction, negation, disjunction, and the square root of negation. The latter is a genuine quantum gate in that it transforms classical registers into quregisters that are superpositions.

The sentential language $Form^{\mathcal{L}}$ is given by negation, conjunction and the square root of negation, with disjunction then defined via the de Morgan law. In quantum computational semantics, the meaning of the linguistic sentences are represented by quregisters.

A quantum computational realization of $Form^{\mathcal{L}}$ is a function Qub from the set of all formulas in $Form^{\mathcal{L}}$ to the union of the Hilbert spaces $\otimes^n\mathbb{C}^2$. The image $Qub(\alpha)$ is called the information value of α. Since $Qub(\alpha)$ is a unit vector of some Hilbert space, it has a probability-value which we use to define a probability-value for α itself. A sentence α is true when its probability-value in a computational realization Qub is 1. The notions of logical truth, consequence, and logical consequence are defined accordingly; the logic characterized by this semantics is called *quantum computational logic* (**QCL**). Distinctions between **QCL** and **QL** are made. It is noted that the flavor of this logic is completely different from that of all the quantum logics that preceded it.

Not addressed herein are the generalized sample spaces of the so called Amherst School, initiated by Foulis and Randall (Foulis, 1999). This theory attempts to generate a logic for any empirical situation, the models for which are precisely the models that we develop for quantum logic. Much of the theory of the algebraic structures that shall concern us was generated by this School.

Nor have we addressed the generalized measure theory that has arisen in which a Boolean algebra of measurable sets is replaced by a more general suitable structure - one of the models that we discuss. While this development may turn out to be of mathematical importance in applications of the theory, it does not directly impact the logic. Nor do we speculate on anticipated applications to the logical underpinnings of other empirical sciences, including biology, economics, political science, and psychology.

What we present is the best that we can do at the moment. We do not present the final definitive way of thinking quantum logically. Although, in most cases, what we do present seems to us natural enough to be lasting. Whatever may come of quantum logics in the future, we hope that a reading of this book will leave no question that quantum logics are (or can be made into) logics in the truest sense of the logical tradition. We hope that we have conveyed the collective enthusiasm of so many of the researchers in this rapidly developing field.

Errors

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