PART I Mathematical and Physical Background

Introduction

In 1920 Łukasiewicz published a two-page article whose title was "On three-valued logic." The paper proposes a semantic characterization for the logic that has been later called $Ł^3$ (Łukasiewicz' three-valued logic). In spite of the shortness of the paper, all the important points concerning the semantics of $Ł^3$ are already there and can be naturally generalized to the case of a generic number n of truth-values as well as to the case of infinite many values. The conclusion of the article is quite interesting:

> The present author is of the opinion that three-valued logic has above all theoretical importance as an endeavour to construct a system of non-aristotelian logic. Whether the new system of logic has any practical importance will be seen only when the logical phenomena, especially those in the deductive sciences, are thoroughly examined, and when the consequences of the indeterministic philosophy, which is the metaphysical substratum of the new logic, can be compared with empirical data (Łukasiewicz, 1970c).

These days, Łukasiewicz' remark appears to be highly prophetic, at least in two respects. First of all, the practical importance of many-valued logics has gone beyond all reasonable expectations at Łukasiewicz' times. What we call today *fuzzy logics* (natural developments of Łukasiewicz' many-valued logics) gave rise to a number of technological applications. We need only recall that we can buy washing machines and cameras whose suggestive name is just "fuzzy logic."

At the same time, quantum theory (QT) has permitted us to compare the consequences of an *indeterministic philosophy* with *empirical data*. This has been done both at a logico-mathematical level and at an experimental level. The so called *no go theorems*[1] speak to the impossibility of *deterministic completions* of orthodox QT by means of *hidden variable theories*. Interestingly enough, some experiments that have been performed in the Eighties[2] have confirmed the statistical predictions of QT, against the predictions of the most significant hidden variable theories.

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[1] See (von Neumann, 1932; Jauch, 1968; Kochen and Specker, 1967; Bell, 1966; Giuntini, 1991a).

[2] See (Aspect, Grangier and Roger, 1981; Aspect and Grangier, 1985).

Łukasiewicz was a contemporary of Heisenberg, Bohr, von Neumann. Strangely enough, however, he very rarely makes explicit references to QT. In spite of this, he seems to be aware of the importance of QT for his indeterministic philosophy. In 1946 he writes a revised version of his paper "On Determinism," an address that he delivered as the rector of the Warsaw University for the inauguration of the academic year 1922/1923. At the very beginning of the article he notices:

> At the time when I gave my address those facts and theories in the field of atomic physics which subsequently led to the undermining of determinism were still unknown. In order not to deviate too much from, and not to interfere with, the original content of the address, I have not amplified my article with arguments drawn from this branch of knowledge (Łukasiewicz, 1970b).

For Łukasiewicz, the basic reason that suggested going beyond classical *bivalent* semantics was a philosophical one.[3] His main argument, developed in the paper "On Determinism" can be sketched as follows:

- **First statement**: *bivalence implies determinism.*
- **Second statement**: *determinism contradicts our basic intuition about necessity and possibility.*
- **Conclusion**: *bivalence has to be refused.*

The argument essentially refers to *temporal sentences* that describe *future contingent events*. A typical example is represented by a sentence like

John will not be at home tomorrow noon.

More formally we can write a temporal sentence as: $\alpha(t)$, to be read as "α is the case at time t." A temporal sentence $\alpha(t)$ may be true or false with respect to a given time t_1, which may either precede or follow t. We will write (according to the usual semantic notation): $\models_{t_1} \alpha(t)$ and $\models_{t_1} \neg\alpha(t)$, respectively.

How can the first statement (*bivalence implies determinism*) be defended? Let us assume the bivalence principle:

any sentence is either true or false (*tertium non datur*).

And let us apply this principle to the case of temporal sentences. Suppose $t_1 < t$. By bivalence we have:

$$\models_{t_1} \alpha(t) \text{ or } \models_{t_1} \neg\alpha(t).$$

In other words, the *event* described by $\alpha(t)$ is already *determined* at time t_1. Suppose: $\models_{t_1} \alpha(t)$. Then, $\alpha(t)$ turns out to describe a *necessary event.* As a consequence, we can conclude that: bivalence implies determinism.

How can we justify the second statement (*determinism contradicts our basic intuition about necessity and possibility*)? Suppose $\alpha(t)$ describes a contingent event (such as the fact asserted by the sentence "John will not

[3] See (Dalla Chiara and Giuntini, 1999).

be at home tomorrow noon"). Now our basic intuition about contingency and necessity seems to require that both sentences $\alpha(t)$ and $\neg\alpha(t)$ must be *possible* at time t_1. As a consequence, future contingent events cannot be determined at any previous time (by definition of contingency).

Strangely enough, from the historical point of view, the abstract researches on fuzzy structures and on quantum structures have undergone quite independent developments for many decades during the 20-th century.

Only after the Eighties, there emerged an interesting convergence between the investigations about fuzzy and quantum structures, in the framework of the so called *unsharp approach to quantum theory*. In this connection a significant conjecture has been proposed: perhaps some apparent *mysteries* of the quantum world should be described as special cases of some more general *fuzzy phenomena*, whose behavior has not yet been fully understood.

The ambiguities of the quantum world can be investigated at different levels. The first level involves the *essential indeterminism* of QT. In order to understand the origin of such indeterminism, from an intuitive point of view, it will be expedient to follow the argument proposed by Birkhoff and von Neumann in their celebrated article "The logic of quantum mechanics" (Birkhoff and von Neumann, 1936). At the very beginning of their paper, Birkhoff and von Neumann observe:

> There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase-space". According to this concept, any physical system $\mathfrak{S}$ is at each instant hypothetically associated with a "point" in a fixed phase-space Σ; this point is supposed to represent mathematically, the "state" of $\mathfrak{S}$, and the "state" of $\mathfrak{S}$ is supposed to be ascertainable by "maximal" observations.

Maximal pieces of information about physical systems are also called *pure states*. For instance, in classical particle mechanics, a pure state of a single particle can be represented by a sequence of six real numbers $\langle r_1, \ldots, r_6 \rangle$, where the first three numbers correspond to the *position*-coordinates, while the last ones are the *momentum*-components.

As a consequence, the phase-space of a single particle system can be identified with the set $\mathbb{R}^6$, consisting of all sextuples of real numbers. Similarly for the case of compound systems, consisting of a finite number n of particles.

Let us now consider an *experimental proposition* $\mathbf{P}$ about our system, asserting that a given *physical quantity* (also called *observable*) has a certain value (for instance: "the value of position in the x-direction lies in a certain interval"). Such a proposition $\mathbf{P}$ will be naturally associated with a subset X of our phase-space, consisting of all the pure states for which $\mathbf{P}$ holds. In other words, the subsets of Σ seem to represent good mathematical representatives of experimental propositions. These subsets have been called by Birkhoff and von Neumann *physical qualities*. According to alternative terminologies, *physical qualities* are also currently called *physical properties* or *physical propositions* or *physical questions* or *physical events*. At this level of analysis, we will simply say *events*. Of course, the correspondence between the set of all experimental propositions and the set of all events will be many-to-one. When a pure state p belongs to an event X, we can say that the system in state p *verifies* both X and the corresponding experimental proposition.

What about the structure of all events? As is well known, the power set of any set gives rise a *Boolean algebra*. And also the set $\mathcal{F}(\Sigma)$ of all *measurable subsets* of Σ (which is more tractable than the full power set of Σ, from a measure-theoretic point of view) turns out to have a Boolean structure. Hence, we may refer to the following Boolean algebra:

$$\langle \mathcal{F}(\Sigma)\,, \cap\,, \cup\,, {}^c\,, \mathbf{0}\,, \mathbf{1} \rangle\,,$$

where:

1) $\cap\,, \cup\,, {}^c$ are, respectively, the operations intersection, union, relative complement;
2) $\mathbf{0}$ is the empty space, while $\mathbf{1}$ is the total space.

According to a standard interpretation, the operations $\cap\,, \cup\,, {}^c$ can be naturally regarded as a set-theoretic realization of the classical logical connectives *and*, *or*, *not*. As a consequence, we will obtain a classical semantic behavior:

- a state p verifies a conjunction $X \cap Y$ iff $p \in X \cap Y$ iff p verifies both members;
- p verifies a disjunction $X \cup Y$ iff $p \in X \cup Y$ iff p verifies at least one member;
- p verifies a negation X^c iff $p \notin X$ iff p does not verify X.

In such a framework, classical pure states turn out to satisfy an important condition: they represent *pieces of information* (about the physical system under investigation) that are at the same time *maximal* and *logically complete*. They are *maximal* because they represent a *maximum of information* that cannot be consistently extended to a richer knowledge in the framework of the theory (even a hypothetical *omniscient mind* could not know more about the physical system in question). Furthermore, pure states are *logically complete* in the following sense: they *semantically decide* any event. For any p and X,

$$p \in X \text{ or } p \in X^c.$$

The semantic excluded middle principle is satisfied.

To what extent can such a picture be adequately extended to QT? Birkhoff and von Neumann observe:

> In quantum theory the points of Σ correspond to the so called "wave-functions" and hence Σ is a ... function-space, usually assumed to be Hilbert space.

As opposed to classical mechanics, QT is *essentially probabilistic*. Generally, pure states only assign probability-values to quantum events. This is strongly connected with the *uncertainty relations*, which represent one of the most significant dividing line between the classical and the quantum case. Let ψ represent a pure state (a wave-function) of a quantum system and let **P** be an experimental proposition (for instance "the spin-value in the x-direction is up"). The following cases are possible:

(i) ψ assigns to **P** probability-value 1;
(ii) ψ assigns to **P** probability-value 0;
(iii) ψ assigns to **P** a probability-value different from 1 and from 0.

In the first two cases, we will say that **P** is *true* (*false*) for the system in state ψ; in the third case, **P** will be *semantically indeterminate*. This constitutes the first level of ambiguity or fuzziness.

As a consequence, unlike classical mechanics, in QT pure states turn out to represent pieces of information that are at the same time *maximal* and *logically incomplete*. Such divergence between maximality and logical completeness is the origin of most logical anomalies of the quantum world.

A second level of ambiguity is connected with a possibly fuzzy character of the physical events that are investigated. We can try and illustrate the difference between two "fuzziness-levels" by referring to a nonscientific example. Let us consider the two following sentences, which apparently have no definite truth-value:

I) Hamlet is 1.70 meters tall;
II) Brutus is an honourable man.

The semantic uncertainty involved in the first example seems to depend on the logical incompleteness of the *individual concept* associated to the name "Hamlet." In other words, the property "being 1.70 meters tall" is a *sharp* property. However, our concept of Hamlet is not able to *decide* whether such a property is satisfied or not. Unlike real persons, literary characters have a number of indeterminate properties. On the contrary, the semantic uncertainty involved in the second example, is mainly caused by the ambiguity of the concept "honourable." What does it mean "being honourable?" One need only recall how the ambiguity of the adjective "honourable" plays an important role in the famous Mark Antony's monologue in Shakespeare's "Julius Caesar." Now, orthodox QT generally takes into consideration examples of the first kind (our first level of fuzziness): events are sharp, while all semantic uncertainties are due to the *logical incompleteness* of the individual concepts, that correspond to pure states of quantum objects. A characteristic of *unsharp QT*, instead, is to investigate also examples of the second kind (second level of fuzziness).

In the following chapters (of Part I) we will try to understand how the essential indeterministic and ambiguous features of the quantum world are connected with the deep mathematical and logical structures of QT. We will first present the axiomatic foundations of sharp QT (Chapters 1-3). The unsharp version of the theory will be developed in Chapters 4-7.