Mutations of Alternative Algebras

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Mutations of Alternative Algebras

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A Pili y Eva

To Karen, Peggy, Jane and Michael

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PREFACE

In recent years, nonassociative algebras, other than Lie algebras, have frequently appeared in other disciplines in sciences and mathematics, and, in particular, in physics. Around 1978, two classes of nonassociative algebras emerged as algebraic models in theoretical physics and mechanics. One class consisted of flexible Lie–admissible algebras, whose introduction was originally due to A.A. Albert in 1948 [**A2**]. The objective was to extend the framework of quantization problems based on associative algebras to a more general one using flexible Lie–admissible algebras [**O1**]. A book detailing the mathematical exposition in this subject was published in 1986 under the title *Malcev–Admissible Algebras* [**My4**].

Another class comprised nonassociative algebras obtained from "mutations" of associative algebras, which originated from a quantized version of generalized Hamilton's equations [**Sa1**]. Thus, the introduction of these algebras arose from an attempt to generalize classical mechanics as well as quantum mechanics. The present book concerns the mathematical side of the theory of this class of algebras. Included in this book are mutations of alternative algebras which are a natural generalization of mutations of associative algebras and which are mainly distinguished by mutations of Cayley–Dickson algebras. This fact provided the title of this book *Mutations of Alternative Algebras*.

Let A be an alternative algebra with multiplication xy over a field F. For fixed elements $p, q \in A$, the (left) (p,q)–mutation $A(p,q)$ of A is defined as the algebra with multiplication $x * y = (xp)y - (yq)x$ defined on the vector space A. If A is associative, then the product $*$ becomes $x * y = xpy - yqx$, which is the original source of $*$. Although mutation algebras are both Jordan–admissible and Malcev–admissible, their structure theory takes on quite different approach from that in the aforementioned book on Malcev–admissible algebras, and that of other well known nonassociative algebras, such as Lie, alternative or Jordan algebras. This is due to the fact that the class of mutation algebras lacks the intrinsic characterization by means of identities because it does not constitute a variety; therefore, it seems impossible to identify a minimal set of identities that defines this class.

In general, the structure of $A(p,q)$ is largely governed by that of the underlying alternative algebra A as well as by the mutation parameters p and q. One of the main tasks in the study of mutation algebras was the search for an appropriate radical which leads to a meaningful structure theory for $A(p,q)$, regardless of the choice of p and q. Recently such a radical was found; namely, there exists the solvable radical $J(p,q)$ of $A(p,q)$ for any artinian alternative algebra A and for arbitrary fixed elements p and q in A.

The primary concern of this book is to present a self-contained and detailed account of the mathematical theory of mutation algebras. The major portion of the material is devoted to the structure theory for mutations of artinian alternative algebras. Therefore, basic structure theory about alternative algebras, which is summarized in Chapter I without proofs, plays a central role throughout this book, in particular, for Chapters III and IV. Virtually all results in the remainder of this book are presented with proofs in considerable detail. Thus, we have conscientiously tried to make the material accessible to nonspecialists in the subject, who wish to become acquainted with mutation algebras and their applications. Owing to the physical origins of these two classes of algebras, another concern was to offer this book as a counterpart to the book *Malcev–Admissible Algebras*, although they have quite different features of structure theory.

The primary objective of Chapter I is to provide background material pertaining to the ensuing chapters. Some basic results on modules, radical and central simplicity for nonassociative algebras are presented with proofs. The structure of composition algebras and alternative algebras is exposited in Section I.3, but we have made no attempt to provide the proofs which can be found in standard texts on nonassociative algebras [**J3**, **Sc3**,**ZSSS1**]. Two identities that arise naturally from the mutation product $*$ are Malcev–admissibility (Lie–admissibility for the associative case) and Jordan–admissibility. In essence, these are the only identities which play important roles for our discussions, although mutation algebras satisfy many other identities (Chapter V). The main concerns in Sections I.4 and I.5 are to present some basic results about Malcev–admissible algebras and Jordan–admissible algebras, including brief historical remarks. Also, included in these two sections are some classical results on Lie and Jordan ideals in a semiprime associative algebra, which are instrumental for later discussions.

Chapter II is the highlight of this book in that we set forth the framework for the structure theory of mutation algebras. For any artinian associative algebra A, the radicals $R(A)$ and $R(A(p,q))$ of A and $A(p,q)$ (in the sense of Section I.1) have the relation $R(A) \subseteq R(A(p,q))$. In general, these radicals are too far apart from each other, and do not seem to shed enough light on the structure of $A(p,q)$. However, it is possible to locate two ideals $R(p,q) \subseteq J(p,q)$, of $A(p,q)$, between $R(A)$ and $R(A(p,q))$, which interplay nicely with $R(A)$ and $R(A(p,q))$, and with the structure of $A(p,q)$ for any mutation parameters p and q. Moreover, it turns out that $R(p,q)$ is nilpotent in $A(p,q)$ and $J(p,q)$ is the solvable radical of $A(p,q)$. There exists another ideal, $R^0(p,q)$ of $A(p,q)$ sitting inside $R(p,q)$, which describes more explicitly the role of p and q.

The general mutation algebra $A(p,q)$ does not satisfy the third–power identity $(x*x)*x = x*(x*x)$ that is implied by well known identities, such as flexibility or power–associativity. In Section II.2 we discuss the relationships between these identities in $A(p,q)$ and conditions for them to be equivalent. Several earlier works on identity problems in $A(p,q)$ are obtained here as special cases from our general results in this section. The simplicity or primeness of $A(p,q)$ implies the same of A. The converse is not true without an additional condition; the investigation of this condition is the main topic in Section II.3. The ideal $R^0(p,q)$ plays a central role for this. Also, included in this section are the relationships among the nucleus, centers and centroids of $A(p,q)$ and A.

Section II.4 is devoted to the structure of $A(p,q)$ for any simple artinian associative algebra A. The principal result is that if A is simple artinian, then there exists a subalgebra S of $A(p,q)$ such that $A(p,q) = S \oplus R^0(p,q)$ and S is isomorphic to a "twisted" mutation algebra. If $p \neq q$, then S is simple. In case $p = q$, it is shown that either $A(p,p)$ is solvable, or $A(p,p)/J(p,p)$ is a semisimple Lie algebra whose commutator algebra is simple. Finally, in Section II.5 we establish the structure theorem: If A is an artinian associative algebra, then $A(p,q)/J(p,q)$ is a direct sum of simple twisted mutation algebras and semisimple Lie algebras cited above. The relationships among the radicals $R(A) \subseteq R(p,q) \subseteq J(p,q) \subseteq R(A(p,q))$ are also investigated in this section.

Chapter III concerns the structure of mutations of alternative algebras. Virtually all results in Chapter II are extended to the alternative case. Since a simple alternative algebra is either associative or a Cayley–Dickson algebra over its center, this chapter is in large devoted to mutations of Cayley–Dickson algebras C. For an artinian alternative algebra A, $A(p,q)/J(p,q)$ is shown to be a direct sum of simple algebras and semisimple Lie algebras. Those simple summands that do not arise from mutations of an associative algebra are of the form $C(p,q)/J(p,q)$. The main objective in Section III.4 is to determine the structure of $C(p,q)$ and $C(p,q)/J(p,q)$.

Chapter IV deals with the automorphism group $\operatorname{Aut} A(p,q)$ and the derivation algebra $\operatorname{Der} A(p,q)$ of $A(p,q)$. The central importance in this chapter is the structure of $\operatorname{Aut} A(p,q)$ and $\operatorname{Der} A(p,q)$ for central simple artinian alternative algebras A. Thus, if A is not associative, then A is a Cayley–Dickson algebra C. For a generalized quaternion algebra Q, the structure of $\operatorname{Aut} Q(p,q)$ and $\operatorname{Der} Q(p,q)$ is more explicitly described (Section IV.5). Similar results are obtained for $\operatorname{Aut} C(p,q)$ and $\operatorname{Der} C(p,q)$ when $p+q$ is invertible (Section IV.6). For associative algebras A and B, where B is prime with a unit element, it is possible to determine all isomorphisms of $A(p,q)$

to $B(a,b)$ in terms of isomorphisms or anti–isomorphisms of A to B, when (a,b) satisfies a certain normality condition (Section IV.3). This plays a principal role for the main results on $\operatorname{Aut} A(p,q)$.

The primary objective in Chapter V is to investigate identities satisfied by mutations of all associative algebras and representations for these mutation algebras. The T–ideal of identities satisfied by the mutation algebras is not only vastly larger than the identities resulting from Lie– and Jordan–admissibilities, but also it is a formidable task to find a generating set of that T–ideal. This situation makes it impossible to study the structure of mutation algebras and its representations within the intrinsic approach based on the theory of varieties. The first part of this chapter is intended to offer a general tool for the treatment of identity problems in mutation algebras. This is done by establishing a gradation for the T–ideal of these identities. All identities of degree ≤ 4 are determined in this gradation, which result in two new identities of degree 4 that are not consequences of Lie– and Jordan–admissibilities. The second part of Chapter V focuses on a particular type of representation, called here a strict representation, that arises naturally from modules for an associative algebra and from Peirce decompositions of a mutation algebra. Such irreducible representations are determined for mutations of any simple artinian associative algebra.

During the writing of this book, it became apparent that the pervading influence in the approach presented here can be attributed to the theory of artinian associative or alternative algebras, rather than those identities stemming from mutation algebras. Therefore, we have taken the view throughout that the material can be more appealing for applications of the theory of noncommutative associative or alternative algebras than a standard theory of nonassociative algebras. We have chosen not to explore the physical side of the theory of mutation algebras, not only because of our desire to keep the text within an algebraic framework but also because of our insufficient competence in the discipline of physical applications related to mutation algebras.

Since the writing of this book was first undertaken in the fall of 1989, the initial draft of the material went through several revisions and improvements, due to on–going research activities in the subject, notably in some Spanish schools of mathematics, in particular, at the University of Zaragoza and the University of Oviedo. Concurrently with our book project, F. Montaner was working on his doctoral thesis in this subject. He generously made his unpublished work available to us, which has played an indispensible role for the final organization of this book.

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