Systems of Nonlinear Partial Differential Equations

Mathematics and Its Applications

Systems of Nonlinear Partial Differential Equations

Applications to Biology and Engineering

by

Anthony W. Leung
Department of Mathematical Sciences,
University of Cincinnati, Cincinnati, Ohio, U.S.A.

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To Soleda, Jason and Jessica

SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, *p*-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

Reaction-diffusion equations, as the name indicates, came from the mathematical modelling of chemical reactions. As a recognized specialism the subject took form and shape in the 1970's and in spite of its relative youth it is now a well developed substantial research field with a host of applications for instance in population dynamics and ecology, where the individual particles tend to be a bit larger, and in reactor engineering. These two application areas are the main ones considered in this volume.

It is also a field in which applications and 'pure' mathematical structures and notions like order, graphs, topological degree interact nontrivially and beautifully. Finally, despite its importance, it is a field that does not yet have many books devoted to it especially at a level of nonsuperspecialists. These are two excellent reasons for welcoming this volume in this series.

Perusing the present volume is not guaranteed to turn you into an instant expert, but it will help, though perhaps only in the sense of the last quote on the right below.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, February 1989

Michiel Hazewinkel

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PREFACE

In the last twelve years, much progress was made in the use of systems of reaction-diffusion equations in the study of a variety of applied topics: ecological systems, fission reactors, chemical reactions and many others. Although several excellent books related to such systems are available, yet numerous useful results in the last twelve years are not readily accessible in a book form for convenient study and reference. In the mean time the need for applications encourages us to enhance the understanding and improve the skill in analyzing such nonlinear systems of parabolic and elliptic partial differential equations.

Several methods had been extremely fruitful in the analysis and are extensively used in this book: (a) Intermediate-value type existence theorem for elliptic system (cf. section 1.4) is valuable for showing the occurence of steady states together with estimates of their sizes; such theorem actually includes the use of Leray-Schauder topological degree. (b) Differential inequalities for parabolic systems are suitable for considering the time stability of steady state solutions (cf. sections 1.3 and 2.3). (c) Upper and lower solutions combined with suitable monotone schemes provide a constructive approach to obtain the existence of solutions for systems (cf. chapter 5); moreover this method is adaptable to numerical approximations (cf. chapter 6). (d) Bifurcation techniques in functional analysis combined with estimations by means of maximum principles provides understanding of structural changes of positive solutions as various parameters varies globally (cf. section 7.4). (e) For large ecological systems with Neumann boundary conditions, the use of Lyapunov functions together with tehniques in graph theory gives extremely keen insight into the interactions between the various components (cf. sections 7.2 and 7.3). (f) Recent results in strongly order-preserving dynamical systems provide a powerful method to analyze the global behavior of solutions of parabolic systems, as time tends to infinity (cf. section 4.6).

All the above methods are carefully explained in the book. One clearly sees how they successfully lead to applicable results in nonlinear elliptic and parabolic partial differential systems related to many ecological interactions and reactor engineering problems. Chapters 2 and 3 contain many recent theorems in the study of prey-predator and competing species under diffusion. Various types of results are considered, involving a variety of assumptions on the population models. Research in such systems are progressing in such a fast pace that it is impossible to include all interesting discoveries in a few short chapters. Scattered throughout chapters 2, 3, 5 and 7 are some recent results which I believe should be useful to future research in this field. These problems had aroused my attention from the viewpoint of ecology as well as mathematics. Chapter 5 futher gives a systematic study of the method of upper and lower solutions, coupled with general monotone schemes recently developed for large elliptic systems. The technique blends beautifully with the analysis of interacting population models. Moreover, there is a section on time-periodic solutions on parabolic systems. Chapter 4 considers reaction difussion systems for reactor engineering. It studies the multigroup neutron fission models, collecting some results obtained by the methods described above. There is also a section on transport systems. The last section presents some recent elegent results in strongly order-preserving dynamical systems. Such systems are applicable to the study of reactor models, genectics, and coupling cooperating and competing species. Chapter 6 is concerned with computational and numerical analysis. It adapts the monotone scheme method to study finite difference systems of equations. Practical procedures as well as convergence theorems are presented. The first part of chapter 7 combines the use of Lyapunov function and graph theory to give very elegant and general results for large Volterra-Lotka type diffusive systems under Neumann boundary conditions. It summarizes the efforts related to many researchers in the last decade. The second part of chapter 7 employs some results in the earlier chapters and some bifurcation techniques in functional analysis to obtain interesting bifurcating solutions in elliptic prey-predator systems. It analyzes the changes in positive solution structures under Dirichlet boundary conditions as the parameters vary globally.

One of the the aims of the book is to gather many useful materials for researchers in reaction-diffusion systems. It is also hoped that by studying applications and pure mathematical methods simultaneously, it is easier to motivate and teach a variety of students the many difficult relevant subjects. I have tried to present the subject in a way which is accessible to advanced undergraduates in mathematics and beginning graduate students. The students are supposed to have a background in advanced calculus together with only some elementary knowledge of differential equations. Hence, the book begins with basic maximum principles in partial differential equations, differential inequalities, introduction of Hölder spaces, Schauder's estimates for solutions of linear scalar equations etc. Although not all the proofs of these preliminary topics are included, those not commonly accessible in standard text books or are involved with techniques which we will employ extensively in later chapters are all presented in detail. The applications should also be understandable to practical researchers who are not fundamentally concerned with the pure mathematics.

The book primarily uses classical solutons in Hölder spaces, and the methods of generalized solutions are not emphasized. Although $W^{k,p}$ estimates are used a couple of times in the proof of convergence of approximate solutions, the thourough understanding of $W^{k,p}$ strong solutions is not absolutely necessary, if one accepts the validity of such estimates. For completeness, such results together with Sobolev's embedding theorem, and some functional analytic bifurcation theorems are included in the appendix. Moreover, with the use of generalized solutions, much of the theorems in this book can be stated in more general terms. Consequently, they can be extended to be applicable to more general practical situations. However, such a task is not our present emphasis.

Some of the materials had been used in several classes in partial differential equations and seminar courses. They had stimulate students to proceed to further work in various directions of their own interests. The range of application of the methods should not be limited only to ecology and reactors, although they are the prime concerns here. The book should be suitable for a two quarters or one semester course in applied mathematics, or for a nonlinear theory part of a sequence of partial differential equations courses. I had regretably omitted many interesting topics in reaction-diffusion systems, for example travelling waves, combustion, free boundary, etc. A treatment of these and other topics is too lengthy, and is beyond the scope of this present manuscript.

I am grateful to many colleagues, students and friends who had visited me at Cincinnati. They include in chronolgical order: Dr. A. Lazer, Dr. D. Clark, Dr. D. Murio, Dr. G. S. Chen, Dr. B. Benjilali, Dr. P. Korman, and Dr. Z. M. Zhou. Their stimulations are valuable in the development of the subject matter of this book. I also wish to thank Miss June Anderson for her typing of most of the manuscript.

Cincinnati, October 1988 Anthony W. Leung