MINIMAX MODELS IN THE THEORY OF NUMERICAL METHODS

To the memory of my mother
Alexandra I. Sukhareva
and my father
Grigory M. Sukharev

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ALEKSEI G. SUKHAREV
Moscow State University, Russia

Translated from the Russian by Olga Chuyan

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PREFACE TO THE ENGLISH EDITION

In the Russian edition published in 1989, this book was called "Minimax Algorithms in Problems of Numerical Analysis". The new title is better related to the subject of the book and its style. The basis for every decision or inference concerning the ways to solve a given problem is the computation model. Thus, the computation model is the epicenter of any structure studied in the book. Algorithms are not constructed here, they are rather derived from computation models. Quality of an algorithm depends entirely on consistency of the model with the real-life problem. So, constructing a model is an art, deriving an algorithm is a science.

We study only minimax or, in other words, worst-case computation models. However, one of the characteristic features of the book is a new approach to the notion of the worst-case conditions in dynamic processes. This approach leads to the concept of sequentially optimal algorithms, which play the central role in the book.

In conclusion, I would like to express my gratitude to Prof. Dr. Heinz J. Skala and Dr. Sergei A. Orlovsky for encouraging translation of this book. I also greatly appreciate the highly professional job of Dr. Olga R. Chuyan who translated the book.

Moscow, January 1992 *A.G. Sukharev*

PREFACE

The topics of computational methods efficiency and choice of the most efficient methods for solving a specific problem or a specific class of problems have always played an important role in numerical analysis. Now optimization of the computerized solution process is a major problem of applied mathematics, which stimulates search for new computational methods and ways of their implementation.

In this monograph, the ways of estimating efficiency of computational algorithms and problems of their optimality are studied in the framework of a general computation model. The main elements of the general model determining specific models of computation are: the functional or the operator to be approximated, which corresponds to the problem being solved; the class of functions reflecting the information we have about the problem; the class of algorithms that can be used for solving the problem; the criterion for estimating efficiency of the algorithm; the optimality concept; and, finally, the specific notion of optimality of an algorithm within the framework of the adopted general concept. Using this general computation model allows various problems of numerical analysis to be treated in a unified way and enables us to answer a number of fundamental methodological questions and to establish some general properties of optimal algorithms.

All specific realizations of the general computation model that are dealt with in this monograph are based on the minimax optimality concept, which was used in applied mathematics as long ago as in the 19th century by P.L. Chebyshev. This concept reflects our aim to obtain the best guaranteed result (say, accuracy of the solution) corresponding to the information we have about the problem.

Optimal algorithms for solving problems of numerical analysis have been studied in a great number of papers. There have also been published several monographs which, alongside dealing with specific problems, provide a foundation for a general theory of optimal algorithms.

The subjects of this book are considerably different from the traditional subjects of computational methods. The major difference is the close attention paid to adaptive (sequential) computational algorithms, the process of computations being regarded as a controlled process and the algorithm - as a strategy of control. This approach allows methods of game theory and other methods of operations research and systems analysis to be widely used for constructing optimal algorithms. Using these methods proves to be very fruitful and leads to a number of results connected to both conventional and new optimality concepts.

The ultimate goal of studying the various computation models we deal with in this book is construction of concrete numerical algorithms admitting program implementation. The central role belongs to the concept of a sequentially optimal algorithm, which in many cases reflects the characteristics of real-life computational processes more fully than the traditional optimality concepts. We work out a general scheme for constructing sequentially optimal algorithms and formulate requirements to the computation model allowing application of this scheme to various problems of numerical analysis.

The monograph consists of five chapters. The first chapter plays a special role. In this chapter we introduce the necessary terminology, construct a general computation model, and discuss in detail the arising methodological problems. It is important to note that the terminology related to optimal algorithms is not completely settled yet. For instance, the "terminal operation of an algorithm" introduced in Section 3 of Chapter 1 corresponds to the "algorithm" from the monograph by J.F. Traub and H. Woźniakowski [80] (see the reference list). The results obtained in Chapter 1 are frequently used throughout the book. Thus, in a sense, this chapter has an introductory character.

The subsequent chapters are devoted to construction of optimal methods for solving specific problems of numerical analysis. We deal with problems of integration, recovering functions from their values, search for the global extremum, solving equations and systems of equations, finding maximin, and multi-criterion optimization. Naturally, every problem is investigated under specific (and sometimes rather restrictive) assumptions on the functional and algorithmic classes, the criteria for estimating efficiency of the algorithms, and the optimality concepts being used, that is, within the framework of a specific computation model. The choice of specific models in the monograph is motivated by our wish to demonstrate the intrinsic abilities of the general model as fully as possible, and also to reflect the characteristic features of real-world problems. Some of the results closely related to the subject of the book (such as the results on optimal search for the extremum of a unimodal function) are omitted for the only reason that they have already been presented both in monographs and textbooks.

Every chapter and every section starts with a short introduction, which outlines the range of problems and sometimes contains references.

The reference list is quite extended and contains a number of papers that are not directly referred to in the text. These papers either deal with some optimality concepts and ways of estimating efficiency of computational methods, or are closely related to some specific problems of numerical analysis studied in the book. At the same time, a number of papers directly connected to the subject of the book and included in the annotated bibliography to the monograph by J.F. Traub and H. Woźniakowski [80] are not mentioned for the lack of space.

A few words about our system for referring to material within the text. Within every chapter, theorems, lemmas, and formulas have double numbers; the first number is that of the section, and the second one is that of the theorem, lemma, or formula itself. A reference to material within the same chapter contains just these two numbers; a reference to material in a different chapter names the chapter, for example: Theorem 5.3 of Chapter 1, formula (3.4) of Chapter 2.

The author is very grateful to the faculty of the Operations Research Department of Moscow State University. The creative atmosphere of the department was of great importance for preparing the book.

GLOSSARY OF SYMBOLS

We only list here some notations of the general character, without mentioning commonly used notations and special notations introduced in the book. The latter are either used right after they have been introduced or supplied with the necessary references.

$\stackrel{def}{=}$ – equal by definition (the quantity being defined may occur both on the left- and right-hand sides of the formula)
$\Rightarrow$ – implication (... implies ...)
$\Leftrightarrow$ – equivalence
$A \subset B$ – the set B contains the set A (this does not rule out the case $A = B$)
$\lfloor t \rfloor$ – entire part of t, i.e., the greatest integer that is not greater than t
$\lceil t \rceil$ – the least integer that is not less than t
$x := a$ – operator assigning the value a to the variable x
$\square$ – end of the proof, end of the statement (if it follows from the previous considerations or is given without the proof), and also end of the remark, example, or definition, etc., with a special heading
$\mathbb{R}$ – numerical line (the set of real numbers)
$\mathbb{R}^n$ – n-dimensional coordinate space
$\langle a, b \rangle = \sum_{j=1}^{N} a^j b^j$ – inner product of the vectors $a = (a^1, \dots, a^N)$ and $b = (b^1, \dots, b^N)$
$A + B = \{c \mid c = a + b,\ a \in A,\ b \in B\}$
$\|a\|_0 = \max_{i=1,\dots,n} |a^i|$
$\|a\|_1 = \sum_{i=1}^{n} |a^i|$
$\|a\|_2 = \sqrt{\sum_{i=1}^{n} (a^i)^2}$
$\lim_{\delta \to a+} \phi(\delta)$, $\lim_{\delta \to a-} \phi(\delta)$ – right-hand and left-hand limits of the function ϕ of a numerical variable at the point a
$x_* = \arg\max_{x \in K} f(x)$ – any point of the global maximum of the function f on the set K: $f(x_*) = \max_{x \in K} f(x)$, $x_* \in K$
$\mathrm{Arg}\max_{x \in K} f(x)$ – set of all the points of the global maximum of f on K