The Dynamic Systems of Basic Economic Growth Models

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The Dynamic Systems of Basic Economic Growth Models

by

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Preface

Two central problems in the pure theory of economic growth are analysed in this monograph: 1) the dynamic laws governing the economic growth processes, 2) the kinematic and geometric properties of the set of solutions to the dynamic systems.

With allegiance to rigor and the emphasis on the theoretical fundamentals of prototype mathematical growth models, the treatise is written in the theorem-proof style. To keep the exposition orderly and as smooth as possible, the economic analysis has been separated from the purely mathematical issues, and hence the monograph is organized in two books.

Regarding the scope and content of the two books, an “Introduction and Overview” has been prepared to offer both motivation and a brief account. The introduction is especially designed to give a recapitulation of the mathematical theory and results presented in Book II, which are used as the unifying mathematical framework in the analysis and exposition of the different economic growth models in Book I. Economists would probably prefer to go directly to Book I and proceed by consulting the mathematical theorems of Book II in confirming the economic theorems in Book I. Thereby, both the independence and interdependence of the economic and mathematical argumentations are respected.

In contrast to linear dynamics systems, which are unified by the powerful superposition principle, there is no dominant principle unifying all nonlinear dynamic systems. Hence, to analyse nonlinear dynamics successfully, it is at first necessary to specify a properly restricted class of dynamic systems. Thus, in short, the restricted class specified by the powerful homogeneity principle is the primary object for the global analyses in these two books. The homogeneity principle, encompassing the superposition principle, has indeed both mathematical and economic implications of major significance. Nevertheless, specific nonhomogeneous dynamic systems are also considered in the treatment of some basic economic growth models.

This monograph is intended for scholars in economics and applied mathematics. It can be used as a basic or supplementary curriculum text in economic and mathematical programmes.

The manuscript was, apart from the last chapter, completed by July 1992, which accordingly is reflected in the bibliographies. It was important also to include in this volume a systematic comparison of planar differential and difference equations.

Many individuals have contributed, each in an important way, to the completion of this treatise.

My interest in the problems of growth theory was inspired by Professor Ragnar Bentzel. Without his encouragement and advice at the early stage, this study would never have been undertaken. I must also acknowledge a great debt to the late Professor Herman Wold for his erudite guidance as teacher and advisor. He accentuated the need for and the importance of “hard theorems in soft sciences”. His thoughts on axiomatization and empirical observation in economics remain in my memory. I am grateful to Professor Bertil Holmlund for his critique, suggestions, and helpful counsel on former versions of the text. Indeed, I am much obliged to the favourable academic environment at the University of Uppsala.

For many valuable discussions and comments through the years, I am much indebted to Mogens Esrom Larsen, University of Copenhagen. From Preben Alsholm at The Technical University of Denmark, I received constructive commentary. I am happy to express a deep-felt appreciation to my brother, Jens Martin Jensen, a physicist, who was always willing to discuss general mathematical issues and help with clarification of many intricate equations.

It is with great pleasure that I express my gratitude to Niels Henrik Børjesson, graduate student at the Copenhagen Business School, who did much of the final typing and endured my numerous changes in the text. He formulated the layout, developed references, and carefully read the manuscript. Because of the technical character of the material and the size of the volume, these tasks involved great responsibility. Without his help, the text in its present form would not have been realized.

Finally, I am beholden to my wife, Kathleen Gail Jensen, for linguistic improvements and for patience and understanding during the long period of production.

In closing, my sincere thanks are conveyed to the editor and staff at Kluwer Academic Publishers for their fine cooperation.

Of course, I am solely responsible for any remaining errors and omissions.

Copenhagen, November 1993

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