Finite and Locally Finite Groups

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Finite and Locally Finite Groups

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PREFACE

This volume contains the proceedings of the NATO Advanced Study Institute on Finite and Locally Finite Groups held in Istanbul, Turkey, 14–27 August 1994, at which there were about 90 participants from some 16 different countries. The ASI received generous financial support from the Scientific Affairs Division of NATO.

INTRODUCTION

A locally finite group is a group in which every finite set of elements is contained in a finite subgroup. The study of locally finite groups began with Schur's result that a periodic linear group is, in fact, locally finite. The simple locally finite groups are of particular interest. In view of the classification of the finite simple groups and advances in representation theory, it is natural to pursue classification theorems for simple locally finite groups. This was one of the central themes of the Istanbul conference and significant progress is reported herein. The theory of simple locally finite groups intersects many areas of group theory and representation theory, so this served as a focus for several articles in the volume.

Every simple locally finite group has what is known as a *Kegel cover*. This is a collection of pairs $\{(G_i, N_i) \mid i \in I\}$, where I is an index set, each group G_i is finite, $N_i \triangleleft G_i$, and G_i/N_i is simple. Moreover, for each finite subgroup H of G there is an index i such that $H \leq G_i$ and $H \cap N_i = 1$. In other words, each finite subgroup of G is contained in one of the simple sections of the cover. If G is countable then the situation is considerably better. Here one can take the index set as the positive integers: $G_1 \leq G_2 \leq \dots$, $G = \bigcup G_i$, and $G_i \cap N_{i+1} = 1$ for all i . This is called a *Kegel sequence*.

The existence of Kegel covers clearly indicates how the analysis of simple locally finite groups is closely linked to the study of finite simple groups and their subgroups. The Kegel covers also tie in with representation theory. A result of Mal'cev, based on the theory of ultraproducts, shows that, if each finitely generated subgroup of G has a faithful representation of degree at most n , then the same is true of G . To illustrate the usefulness of this result, assume we are given a Kegel cover such that the simple factors have a faithful representation of bounded degree. Every finite subgroup of G is isomorphic to a subgroup of one of the simple factors, so Mal'cev's result implies G has a nontrivial faithful finite dimensional representation. That is, G is *linear*.

The lead article of Hartley provides a survey of various topics in the theory of simple locally finite groups and serves as a more complete introduction to the rest of the volume. The first section concerns constructions and introduces the linear groups, finitary groups, universal groups of P. Hall, and some recent groups due to Meierfrankenfeld. These topics are each covered individually in subsequent articles. The second section consists of a discussion of Kegel covers, leading to an analysis of how the covers affect the global structure of simple locally finite groups. The final section concerns centralizers, difficulties resulting from the existence of Kegel kernels, and some open problems.

Following the classification of finite simple groups, it was shown in the 1980s that the linear simple locally finite groups are groups of Lie type over locally finite

fields. These are the groups that are most closely linked to finite groups. Indeed, such a group is the union of groups of Lie type of a fixed type and the structure and representations of such groups are closely connected with the structure and representation theory of the finite groups of Lie type.

The articles of Liebeck, Saxl, and Seitz are a package aimed at a general understanding of algebraic groups, finite groups of Lie type, subgroups, and representations. The finite groups of Lie type and the simple linear locally finite groups arise naturally from algebraic groups. It has long been recognized that to establish deep results about finite groups of Lie type it is first necessary to understand algebraic groups. Consequently this series of articles begins with Seitz's article on algebraic groups.

The first section of this article proceeds from the definition of algebraic varieties to the classification of simple algebraic groups. This is followed by a discussion of the subgroups of algebraic groups, leading to the determination of the maximal connected subgroups of simple algebraic groups. The final section discusses representation theory, concluding with an application of a new result on abstract homomorphisms to the theory of locally finite groups.

The article of Liebeck presents the general method, based on Lang's theorem, in which one uses information about simple algebraic groups to obtain information about the finite groups of Lie type. Many examples are given. Liebeck also discusses the subgroups of finite simple groups. Included is an outline of a new proof of Aschbacher's reduction theorem on finite classical groups, using descent from algebraic groups. The article concludes with a survey of the current situation on the subgroups of finite exceptional groups, including a reduction theorem analogous to the result for classical groups.

Saxl discusses permutation groups, in particular primitive groups. Included is an outline of the O'Nan-Scott theorem, a reduction theorem for analyzing primitive permutation groups. This result leads one to the study of maximal subgroups of almost simple groups. The article continues with a brief survey of the current state of affairs in the study of maximal subgroups of finite simple groups.

After the linear locally finite groups, the next class of groups to discuss are the *finitary groups*. Let V be a vector space. An element $g \in \text{GL}(V)$ is said to be finitary if $\dim(1 - g)V < \infty$. The collection of all such elements is a group, $\text{FGL}(V)$. A finitary group is a subgroup of this group. Of course, every linear group is finitary. Further examples are fairly easy to construct. Let V have countable basis $v_1, v_2, \dots$. For each i , let $V_i = \langle v_j \mid 1 \leq j \leq i \rangle$ and let G_i denote the subgroup of $\text{GL}(V)$ inducing $\text{SL}(V_i)$ on V_i and the identity on $\langle v_{i+1}, v_{i+2}, \dots \rangle$. Then $G = \bigcup G_i$ is a finitary group, the *stable special linear group on V* . If the base field is locally finite, then G is a simple locally finite group. The special nature of these groups is reflected in the embedding of pairs of terms in the given Kegel sequence—the smaller element of any pair is embedded in the larger in as simple a manner as possible. Another source of examples is provided by alternating groups on an infinite set, using the permutation representation.

The articles of Phillips and Hall discuss these groups in detail. Phillips presents a general survey of finitary groups, not necessarily requiring the group to be locally finite. The article of Hall discusses his recent classification of the finitary simple

locally finite groups. Such a group is shown to be similar to those described above: either alternating or a finitary classical group of specific type. The analysis provides a lovely example of how one can work with Kegel covers with the goal of a global classification.

The articles of Meierfrankenfeld and Belyaev each provide further direction for classification results in the theory of locally finite groups. Belyaev discusses the theory of inert subgroups of locally finite groups. These are subgroups for which the intersection of any two conjugates has finite index in each. Nontrivial examples are always present in countable locally finite simple groups which are not linear.

Meierfrankenfeld presents an important new result on the structure of an arbitrary locally finite simple group. Roughly speaking this result shows that a simple locally finite group fits into one of three categories. The first of these consists of the finitary groups. The groups in the next category are those of *alternating type*, which means that there exists a Kegel cover all of whose factors are alternating groups. The third category consists of groups where there is a Kegel cover $\{(G_i, N_i) \mid i \in I\}$ such that, for each i , G_i/N_i is a projective special linear group over a field of characteristic p and $G_i/O_p(G_i)$ is a product of perfect central extensions of classical groups over fields of characteristic p . Groups of the first type have now been classified and Meierfrankenfeld raises the question as to whether groups of the third type are amenable to classification.

The article of Zalesskii concerns the ideal structure of group rings of simple locally finite groups. Connections are explored between various types of groups, their local systems, and the existence of proper ideals of group algebras other than the augmentation ideal. For example, if G is a linear, simple, locally finite group and F a field of characteristic 0, then FG has no proper ideals other than the augmentation ideal.

The study of groups of finite Morley rank lies on the interface between model theory and group theory. Here the outstanding open problem is the Cherlin-Zil'ber conjecture that a simple group of finite Morley rank is an algebraic group over an algebraically closed field. It was shown by S. Thomas that the conjecture holds if the group is also locally finite. The argument makes heavy use of the classification of finite simple groups. In his article, Borovik gives a general discussion of the theory of simple groups of finite Morley rank and indicates how arguments from the classification of finite simple groups, rather than the classification itself, might yield an approach to the conjecture.

Leinen's article gives a lengthy and detailed survey of existentially closed groups in certain classes. Emphasis is placed on classes of nilpotent groups, locally finite groups, and extensions.

The articles of Bryant, Isaacs, and Turull are each concerned with the representation theory of finite groups. Representation theory of finite groups has several connections with the theory of locally finite groups. For example, Mal'cev's theorem ties the representation theory of the finite groups to that of the whole group, and the Kegel kernels often yield representations of the simple factors.

Turull's article discusses length problems in the theory of finite solvable groups. In particular, connections are described between the nilpotence length of a finite solvable group and information about the group of fixed points of a subgroup of

the automorphism group. The article includes discussions of fixed-point-free actions and the existence of regular orbits. Issues of this sort have connections with the difficult problem of analyzing centralizers in simple locally finite groups. For example, Hartley's survey article includes an application of Dade's famous result on Carter subgroups of finite solvable groups to the theory of centralizers in locally finite groups.

The article of Bryant concerns the representation theory of $GL_n(K)$, for K an infinite field. There is a natural action on the polynomial algebra $K[x_1, \dots, x_n]$ in which the homogeneous polynomials of a given degree form an invariant subspace. When K has characteristic 0, these subspaces are each irreducible and afford the appropriate symmetric power of the natural representation. The situation is much more complicated in positive characteristic. Bryant presents an account of results of Doty and Krop which describe the submodule structure. The second part of the article gives a new proof of the result that $K[x_1, \dots, x_n]$ is asymptotically close to being free when regarded as a module for a finite subgroup of $GL_n(K)$.

Isaacs discusses the modular representation theory of finite solvable groups; in particular, generalizations of the Brauer theory to sets of primes. For a set π of prime numbers, the π -partial characters of a finite group G are the restrictions of the ordinary characters of G to the π -elements of G . Isaacs describes an extensive theory of π -partial characters of solvable groups, which can be used to establish results about ordinary characters. Applications are given to the theory of monomial groups and groups of odd order.

The volume concludes with the article of Shalev, which presents a fascinating discussion of recent results in the theory of finite p -groups and their inverse limits, particularly those topics which are closely related to Lie theory. Special topics include a discussion of *powerful p -groups* (p -groups where all p -th powers lie in the derived group) and their connection with the theory of p -adic analytic groups, and also a discussion of Zelmanov's work on the restricted Burnside problem. The article includes a large collection of open problems.

On October 8, 1994, Brian Hartley died suddenly while hiking. The Istanbul conference was from the beginning Brian's idea and he spent an enormous amount of time planning the conference. His vision was a conference where mathematicians from different areas would come together and interact in a nontrivial way. The theory of finite and locally finite groups proved to be a perfect subject for this, and the conference was a huge success. Beyond this, Brian wanted to provide an environment for the education of graduate students—lectures should start at the elementary level and build to recent advances. The format of a NATO Advanced Study Institute was ideal for what he had in mind.

The tone of the conference is clear from the articles in this volume. Brian brought together a collection of participants many of whom rarely see each other. Yet everybody seemed to fit at the conference and attendance at talks rarely dropped off. The special nature of the conference was clear from the beginning and Brian's influence was always present.

This volume is dedicated to the memory of Brian Hartley.

Gary M. Seitz