

FOUNDATIONS OF CODING THEORY

EPISTEME

A SERIES IN THE FOUNDATIONAL,
METHODOLOGICAL, PHILOSOPHICAL, PSYCHOLOGICAL,
SOCIOLOGICAL AND HISTORICAL ASPECTS
OF THE SCIENCES, PURE AND APPLIED

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FOUNDATIONS OF CODING THEORY

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PREFACE

During the sixteenth century, Cardano wrote a fascinating work called *The Book on Games of Chance*. In it he gives an extremely candid recounting and personal appraisal of some aspects of his most remarkable life.* One feature of the book is striking for the modern scientist or mathematician accustomed to current publishing practices. It is brought out during Cardano's discussion of his investigations of certain special questions of applied probability, namely, the question of how to win at gambling. His technique is simplicity itself: in fine reportorial style he reveals his proposed strategy for a particular gambling game, giving marvelous motivating arguments which induce the reader to feel warm, heartfelt support for the projected strategy. Then with all the drama that only a ringside seat observation can bring, Cardano announces that he tried the strategy at the casino and ended up borrowing his taxi fare. Undaunted by failure, he analyzes his now fire-tested strategy in detail, mounts new and persuasive arguments, and, ablaze with fresh optimism and replenished resources, charges off to the fray determined to now succeed where he had so often failed before.

Along the way, Cardano developed a number of valuable insights about games of chance and produced useful research results which presumably would be of interest in our present-day society. However, he could never publish the results today in journals with all the flair, the mistakes, the failures and minor successes which he exhibits in his book. Journal editors rarely give page space to a recounting of mistakes, no matter how charmingly told. They are even reluctant to publish papers which seek to provide a developmental account of the thinking which eventually led to the results being reported. All of which seems remarkable in view of the simultaneous efforts to attract and train new workers in the field: one almost has the impression they feel that the proper way to equip

* A translation by Sidney Gould of this work on gambling by Cardano appears in *Cardano, The Gambling Scholar* by Oystein Ore, Princeton University Press (1953).

the neophyte worker is to shield him from the work of others and provide him with only the "latest" research results of such work.

Any attempt to report the work of a group of people presents difficulties, particularly if the work has been carried out over a period of time with shifts in group membership. The usual outcome of such attempts is either a polished and somewhat sterile version of the final results or a more or less personal history of the group's work habits: the flavor of individual contribution and the benefit of a developmental overview are lost. In the first case, the emphasis is on the results of the group; in the second, on the group to the detriment of its findings. In rare cases, an opportunity exists to present important technical results obtained by a group in a fashion which makes clear how the concepts developed and yet maintains the individuality of the contributors.

Such is the case with the Coding Group at Parke Mathematical Laboratories, Incorporated in Carlisle, Massachusetts, U.S.A., which was occupied with various problems in Coding Theory from 1957 until 1968 when the work terminated. Membership in the Group changed during the eleven years and the range of effort varied. The last four years of this period saw a concerted and systematic attack on the basic problems of Coding Theory in a context more general than that which had been reported in the literature. What emerged from these efforts has been termed Abstract Coding Theory and this book is devoted to an exposition of the foundations of that theory.

Five individuals were concerned with this study: Lorenzo Calabi who directed the effort, John Riley, Lester Arquette, Theodore Hatcher and the writer, William Hartnett. In a long series of some twenty-seven separate and joint papers and memoranda they explored the properties of codes, formulated theoretical foundations for a study of Abstract Coding Theory, fashioned important tools for the study, and constructed specific families of codes with desirable properties. A few but not all of the papers have been reported in the literature. Our aim here is to organize ten of the more important of these papers into a sequence of chapters which reflects the temporal development of the concepts studied and which provides an intellectual framework for an understanding and appreciation of the generality of the approach used and of the results obtained. The book is divided into three parts: the first is introductory and mainly expository, the second presents the extensive theoretical

work on abstract codes, and the third provides tests of code properties and certain constructions of families of codes. The first chapter, separately written by the editor, introduces the four basic problems dealt with in the later chapters and provides a general background for the papers. The overviews for Parts I and II summarize the basic content of the parts. The editorial notes which precede the chapters describe the historical and conceptual framework in which each paper was written. A reference of the form [PML 21] refers to item 21 of the Parke Selected Bibliography; [21] would refer to the usual references.

It seems worthwhile to comment briefly on the composition of the group because of its influence on the development of Abstract Coding Theory. With the exception of Arquette whose basic training was in electrical engineering with secondary work in mathematics, all members of the Group were pure mathematicians applying mathematics, its conceptual framework and its arsenal of techniques, to the problems of Abstract Coding Theory. With contractual affiliations with communication engineers at Air Force Cambridge Research Laboratories, Bedford, Massachusetts, the Group functioned with a foot in each of two worlds. Living and working in the world of mathematics, the Group maintained a constant awareness of the differences in approach of the two worlds and made consistent and strenuous efforts to relate its results with the needs and interests of a broader audience. In practice, this meant that care was given in the exposition of the work to make a mathematical approach to the problems meaningful and attractive. It would be pleasant to report that our efforts were unqualified successes but we are not at all sure that such is the factual situation. However, even modest success would amply reward our efforts. After all, we did enjoy the work. It is our fond hope that others may find some pleasure in reading this account of our work.

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for the PML Coding Group

Plattsburgh, New York, U.S.A.
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W. E. H.

CROSS REFERENCES FOR THE CHAPTERS IN
THE PARKE SELECTED BIBLIOGRAPHY

Note: The versions of the papers in this book are sometimes slightly edited versions of the originals.

Chapter 2 = PML35

Chapter 3 = PML43

Chapter 4 = PML44

Chapter 5 = PML46

Chapter 6 = PML47

Chapter 7 = PML59

Chapter 8 = PML34

Chapter 9 = PML42

Chapter 10 = PML55

Chapter 11 = PML48

Numbered references refer to the general bibliography.