Rings and Geometry

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Rings and Geometry

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TABLE OF CONTENTS

PREFACE

ix

PART I: NON-COMMUTATIVE ALGEBRAIC GEOMETRY

P.M. COHN / Principles of non-commutative algebraic geometry	3
§ 1 Free algebras and free fields	4
§ 2 Specializations and the rational topology	13
§ 3 Singularities of matrices over a free ring	18
§ 4 Existentially closed fields and the Nullstellensatz	26
H. HAVLICEK / Applications of results on generalized polynomial identities in Desarguesian projective spaces	39
§ 1 Introduction	39
§ 2 Non-degenerate normal curves	45
§ 3 Degenerate conics	62
§ 4 Degenerate normal curves	73

PART II: HJELMSLEV GEOMETRIES

J.W. LORIMER / A topological characterization of Hjelmslev's classical geometries	81
§ 1 Hjelmslev planes and Hjelmslev rings	83
§ 2 Construction of commutative H-rings	89
§ 3 The geometric significance of nilpotent radicals	96
§ 4 Topological Hjelmslev planes	99
§ 5 Locally compact H-planes	116
§ 6 Characterizations of commutative H-rings with nilpotent radicals	123
§ 7 Locally compact connected pappian Hjelmslev planes	127
D.A. DRAKE and D. JUNGnickel / Finite Hjelmslev planes and Klingenberg epimorphisms	153
§ 1 K-structures and H-structures	155
§ 2 Nets and non-existence results for K-structures	159
§ 3 Nets and non-existence results for H-structures	163
§ 4 Desarguesian K-planes	168
§ 5 Auxiliary matrices	177
§ 6 Quadratic forms and a PH-plane with $q_2 \neq q_1$	185
§ 7 Regular K-structures	191
§ 8 Generalizations of Singer's theorem and a recursive construction	198
§ 9 Eumorphisms of regular K-structures	204

§ 10	Balanced H-matrices	209
§ 11	Recursive constructions	216
§ 12	Open problems	223

PART III: GEOMETRIES OVER ALTERNATIVE RINGS

J.R. FAULKNER and J.C. FERRAR / Generalizing the Moufang plane	235
§ 1 Inhomogeneous and homogeneous coordinates	236
§ 2 Collineations of real projective planes	238
§ 3 The real projective plane $P(\mathbb{R})$ as a homogeneous space	240
§ 4 Abstract projective planes	242
§ 5 A Jordan algebra construction of projective planes	251
§ 6 The Hjelmslev-Moufang plane	259
§ 7 Algebraic transvections in $P(0)$	262
§ 8 Axiomatization and coordinatization of $P(0)$	268
§ 9 $P(0)$ as homogeneous space	273
§ 10 Another realization of Σ	276
§ 11 Jordan pairs - a final look at the Hjelmslev-Moufang plane	279
§ 12 Abstract Moufang-Veldkamp planes	284
F.D. VELDKAMP / Projective ring planes and their homomorphisms	289
A. Algebraic preliminaries	
§ 1 Free modules and their subspaces	291
§ 2 Stable rank of a ring	292
B. Projective ring planes	
§ 3 The projective plane over a ring of stable rank 2	300
§ 4 Barbilian planes	304
§ 5 Collineations and affine collineations	306
§ 6 Barbilian transvection planes	313
§ 7 Projective ring planes	316
§ 8 Coordinatization of projective ring planes	317
§ 9 Projective planes over special types of rings	321
C. Homomorphisms of projective ring planes	
§ 10 Homomorphisms of Barbilian planes	326
§ 11 Distant-preserving homomorphisms	335
§ 12 Algebraic characterization of full incidence homomorphisms	340
§ 13 Full neighbor-preserving homomorphisms	342
§ 14 Admissible subrings	342

PART IV: METRIC RING GEOMETRIES, LINEAR GROUPS OVER RINGS AND COORDINATIZATION

C. BARTOLONE and F. BARTOLOZZI / Topics in geometric algebra over rings	353
§ 1 Collineations between projective spaces	354
§ 2 Collineations between lines	356

§ 3	Non-injective maps which preserve generalized harmonic quadruples	368
§ 4	The structure of $GL_n(R)$	373
B.R. McDONALD / Metric geometry over local-global commutative rings		391
§ 1	LG-rings	394
§ 2	Linear algebra	396
§ 3	$GL(2)$	398
§ 4	Inner product spaces and the orthogonal group	401
§ 5	Witt rings	406
§ 6	The symplectic and unitary groups	409
B.R. McDONALD / Linear mappings of matrix rings preserving invariants		417
§ 1	Introduction	417
§ 2	The linear algebraic approach of McDonald, Marcus, and Moyls	420
§ 3	The group scheme approach of Waterhouse	427
§ 4	Concluding remarks	433
H. KARZEL and G. KIST / Kinematic algebras and their geometries		437
§ 1	Motivation and historical review	437
§ 2	Problems resulting from classical kinematics; a survey of the material covered in this paper	451
§ 3	2-algebras	452
§ 4	Alternative kinematic algebras	461
§ 5	Geometric derivations of 2-algebras	479
§ 6	2-algebras whose projective derivation is an affine porous space	481
§ 7	The kinematic derivation of an alternative kinematic algebra. Representation theorem	484
§ 8	Kinematic algebras with an adjoint map. The general notation of a kinematic map	490
§ 9	Kustaanheimo's kinematic model of the hyperbolic space	502
U. BREHM / Coordinatization of lattices		511
§ 1	Introduction	511
§ 2	Basic definitions and notations	514
§ 3	The axioms and formulation of the coordinatization theorem	516
§ 4	Lemmata	518
§ 5	Proof of the coordinatization theorem	527
§ 6	A different approach	539
§ 7	The independence of the axioms	544
	Appendix	546

EPILOG

C. ARF / The advantage of geometric concepts in mathematics 553

INDEX OF SUBJECTS 557

PREFACE

When looking for applications of ring theory in geometry, one first thinks of algebraic geometry, which sometimes may even be interpreted as the concrete side of commutative algebra. However, this highly developed branch of mathematics has been dealt with in a variety of monographs, so that - in spite of its technical complexity - it can be regarded as relatively well accessible. While in the last 120 years algebraic geometry has again and again attracted concentrated interest - which right now has reached a peak once more - , the numerous other applications of ring theory in geometry have not been assembled in a textbook and are scattered in many papers throughout the literature, which makes it hard for them to emerge from the shadow of the brilliant theory of algebraic geometry.

It is the aim of these proceedings to give a unifying presentation of those geometrical applications of ring theory outside of algebraic geometry, and to show that they offer a considerable wealth of beautiful ideas, too. Furthermore it becomes apparent that there are natural connections to many branches of modern mathematics, e.g. to the theory of (algebraic) groups and of Jordan algebras, and to combinatorics. To make these remarks more precise, we will now give a description of the contents.

In the first chapter, an approach towards a theory of non-commutative algebraic geometry is attempted from two different points of view. Firstly through generalization of the algebraic tools, which involves in particular a study of the non-commutative analog of the function field of an algebraic variety; secondly through the use of synthetic geometry, many of whose definitions for classical objects of algebraic geometry (such as rational normal curves) carry over to the case of a non-commutative coordinate field.

The second chapter is dedicated to the more traditional theory of projective Hjelmslev planes, where homomorphisms rather than single planes lie in the center of interest. First the emphasis is put on topological and particularly locally compact Hjelmslev planes and their topological properties; also the connection between the algebraic properties of topological coordinate rings and the incidence-theoretic behaviour (closedness of configurations) of the associated affine Hjelmslev planes is investigated. Then for finite Hjelmslev planes the factorization problem is studied in detail, and a number of existence- resp. nonexistence theorems are derived from this. Some fruitful construction principles for finite Hjelmslev planes are presented, e.g.

the method of auxiliary matrices, and influences on the theory of designs are pursued.

The third chapter has its historical origins in the investigations of H. Freudenthal about Moufang planes and in the papers of T.A. Springer and F.D. Veldkamp about Hjelmslev-Moufang planes. The aim of these papers was to establish a bridge between geometry on one side and the theory of Jordan algebras and algebraic groups on the other side. In the first part of this chapter, the classical construction of a Hjelmslev-Moufang plane over any octonion algebra (by means of a reduced exceptional simple Jordan algebra) is modified in order to yield a method of constructing so-called "Moufang-Veldkamp planes" whose coordinate ring can be any alternative ring of stable rank 2; these "Moufang-Veldkamp planes" are defined abstractly by a set of axioms on a plane incidence structure which is additionally endowed with a neighborhood relation. Furthermore, a proof is indicated for the fact that every Moufang-Veldkamp plane arises in the above way. The second part of the chapter summarizes the axiomatic theory of projective planes over associative rings of stable rank 2; apart from general assumptions on the incidence- and neighborhood-relation, the existence of sufficiently many transvections, dilatations and their duals must be required in order to characterize these planes. Finally, the homomorphisms between two such ring planes are classified.

The fourth chapter begins with an account of the possibilities of extending the structural identity between classical projective geometry and linear algebra over division rings to arbitrary rings; over large classes of rings the structure of special linear, orthogonal, symplectic and unitary groups is investigated and the isomorphisms between such groups are studied. When classifying the linear maps of matrix rings (over commutative rings) that preserve certain invariants (e.g. the determinant), unexpected connections with the theory of affine group schemes become apparent.

One contribution of this chapter is dedicated to the geometrical applications of quadratic algebras (which are called kinematic algebras there); in particular, the geometrical structure of the units of alternative quadratic algebras is put in contrasting perspective to the third chapter.

The fourth chapter ends with a discussion of how to find natural coordinatizations of general geometries. When taking a lattice as a prototype of such a geometry, then coordinatization in a very general sense means a construction of an algebraic structure (mostly a module) whose lattice of substructures or congruence-relations is isomorphic to the given lattice. By simple, natural and independent axioms of a geometric flavor lattices are characterized which are isomorphic to submodule lattices of torsion free modules of Goldie dimension at least three over left Ore domains.

An appendix to the book contains an encouragement to the geometers by C. Arf, where he describes the concepts of geometry as indispensable for mathematics.

We are very grateful to the NATO Science Division for providing the funds for this Advanced Study Institute and to the participants in this summer school who were a receptive audience for the lectures. But primarily we wish to thank the authors who strove to keep a balance between their present preferences and the prescribed overall theme. We hope that thereby a book has come into being which -in spite of the personal preferences of the authors which have not been suppressed- reveals a unifying approach. Thus it can well serve as a substitute for a lacking monograph in that area of mathematics where the ring theory occuring outside of algebraic geometry is harmonically unified with geometry.

Erlangen and Istanbul,

May 31, 1985

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