Unbounded Non-Commutative Integration

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Unbounded Non-Commutative Integration

by

J. P. Jurzak

Physique-Mathématique, Université de Dijon, France

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PREFACE

Non-commutative integration has its origin in the classical papers of Murray and von Neumann on rings of operators, and was introduced because of unsolved problems in unitary group representations and the elucidation of various aspects of quantum-mechanical formalism, together with formal calculus in such operator rings. These papers emphasized the interest in II_1 -factors and pointed out the remarkable behavior and algebraic structure of the set of all unbounded closed operators affiliated to such rings. The absence of power tools in functional analysis - mainly settled in their definitive form by A. Grothendieck around 1950-1955 - together with the pathological manipulation of algebraic operations on closed operators in Hilbert spaces, has limited ring-theory to the study of algebras of bounded operators with the main objective the difficult question of classification up to isomorphisms of factors.

This material has permitted a rigorous study of discrete systems in statistical mechanics but appears to be less convincing in other domains of physics (in the algebraic approach to field theory, for example). The striking role of Hamiltonians, Schrödinger operators and Lie group invariant properties in such areas of physics disappears in the so-called C^* -approach. One has to note here that algebras of bounded operators may efface certain features: in this respect we find, for a space A of observable containing all powers Δ^n of a given essentially self-adjoint hamiltonian Δ , assuming that Δ^{-1} is compact, that states on A are associated to particular trace class operators acting in the Hilbert space H - even if A contains the space $L(H)$ of all bounded operators in H - i.e., the predual of A is identical to its strong dual. Self-adjointness of hamiltonians is a natural physical property, though we know that general symmetric observables may have zero self-adjoint extension. Finally, the presence of quadratic forms seems unavoidable in these theories and has been already introduced by certain authors.

However, many mathematical problems in Hilbert spaces cannot be reduced to bounded operators problems, even with the use of bounded functions (when defined?) of unbounded operators. It is significant that Sobolev spaces, or related spaces of partial differential equations, are constructed from self-adjointness on C_c^∞ of a suitable dominating partial differential operator (one may take all powers of the

Laplacian) and these spaces of regular vectors lead to the manipulation of the supple space of differentiable vectors for a self-adjoint operator or, more generally, of the space of C^∞ -vectors for a Lie group representation canonically present. A common denominator in these situations is the presence of a Fréchet space topology on these domains of regular vectors, which dictates the behavior of these dominating operators and explains the methods of estimations a priori. Such Fréchet spaces are often Schwartz spaces or nuclear spaces of C^∞ -functions (a consequence of the Rellich theorem and Sobolev lemma) and correspond to the discrete nature of the spectra of the operators mentioned. These techniques also work in enveloping algebras obtained from the differentiation of a Lie group representation, where domination is ensured by the image of the Laplacian constructed on this group G . Finally, the Plancherel formula on G is a situation involving the distribution at the neutral element and cannot be treated with bounded operators only.

The fusion of all the preceding ideas and situations has led the author to modify the ring-operator theory in order to make it applicable to these rough contexts. This monograph is devoted to the study of the indispensable tools needed to accomplish this. The analogue of the Hamiltonian or of the Laplacian must be a fixed self-adjoint operator Δ whose property is that all its powers Δ^n are dominating in the space considered. The material so obtained is different, and not equivalent, to von Neumann's theory, though essentially the result of von Neumann investigations.

The author thanks Professors H. Araki and M. Flato for their kind interest in this work and valuable comments.

INTRODUCTION

This work develops the material for unbounded non-commutative integration and describes the relation between these developments and material in analysis concerned with regularity questions. Unbounded operators automatically appears in such questions and their manipulation is known to be delicate in limit processes, commutation questions, etc.

On the other hand, traditional non-commutative integration is a well-established theory dealing with algebras of bounded operators and has been distant from classical analysis for a long time; recently, however, important applications have been developed by A. Connes (see [5]).

The commutative case is clearly concerned with functional integration and makes the use of unbounded measurable functions transparent. In an unbounded non-commutative case, they are replaced by continuous sesquilinear forms on a given dense domain $\mathcal{D}$ - endowed with a natural Fréchet topology - dense in some Hilbert space H ; note that a closeable operator T in H with a domain containing $\mathcal{D}$ is viewed as the continuous sesquilinear form $(x, y) \in \mathcal{D} \times \mathcal{D} \rightarrow (Tx, y)$. Elements of the dual or antidual of $\mathcal{D}$ are sometimes called distributions, as it is clearly seen in example 6. A known example is $\mathcal{D}$ equal to the Fréchet space of C^∞ -vectors for a Lie group representation. The usual Banach algebra $L(H)$ is replaced - except in commutation questions - by the set $B(\mathcal{D}, \mathcal{D})$ of all continuous sesquilinear forms β on the product space $\mathcal{D} \times \mathcal{D}$, and, as $B(\mathcal{D}, \mathcal{D})$ is the dual of the (Fréchet space) projective tensor product $\mathcal{D} \hat{\otimes} \mathcal{D}$ (isomorphic to the predual of $B(\mathcal{D}, \mathcal{D})$), we see how smooth the limit process is for sequences or nets in $B(\mathcal{D}, \mathcal{D})$.

The concept of a von Neumann algebra is replaced by the concept of an ultraweakly closed space A with condition II; in general, A is not an algebra and one may consult the paragraph on examples and comments below. The bounded part A_{id} (i.e. the set of β in $\mathcal{D} \times \mathcal{D}$ which extends continuously to $H \times H$) is a von Neumann algebra contained in A practised from many points of view.

The strong dual A^β and the predual space P_A of A are always Fréchet spaces and the space A is endowed with an inductive limit topology - called ρ - of normed spaces A_{A_1} constructed from domination with a given $A_1 \geq \text{Id}$ in A ; this domination is closed to Nelson's domination in [15] and to generalized strong convergence studied in [12]. The passage

from A_{A_i} to A_{id} is ensured by an easy lift process corresponding to a homeomorphism of A . This paper is a continuation of [1, 10] and has been written mainly in a self-contained manner.

The aim of Chapter 1 is to make the reader familiar with ultraweakly closed spaces (with condition II) and commutation theory inside $B(\mathcal{D}, \mathcal{D})$, which is sufficient for many questions. Our point of view is slightly different from [1] - though proof of theorem 1.2 is directly adapted from this paper - and try to present an ultraweakly closed space A as a natural object existing alone independently of any $\star$ -algebra in it. Note also that excessive generality may be avoided by considering those spaces A with a cofinal abelian sequence (see also examples 2, 4 and remark 1.1.1°/). We also see that A may be viewed as the space of elements β in $B(\mathcal{D}, \mathcal{D})$ affiliated to the von Neumann algebra $M = A_{id}$.

Chapter 2 essentially establishes the density of M for ρ -topology considered on A .

Chapter 3 concerns the technical results on elementary operations (reduction, induction, etc.) as a preparation for Chapter 10.

Chapter 4 is devoted to a detailed analysis of the Fréchet space $\mathcal{D}$ its strong dual $\mathcal{D}'$ with a supplementary section on the manipulation of closed operators in Hilbert spaces. The construction of the domains necessary for a general commutant theory is indicated (compare with example 3).

Chapter 5 describes the abelian case. Using remark 1.1.1°/, we are reduced to $A = \bigcup_{n \geq 0} A_{\Delta^n}$, $\Delta \geq 1$: the image of

the Gelfand transformation is explicit and the set X of characters of A is a dense open set of the compact space K (K is the space of characters of A_{id}) thus representing elements of A as continuous functions on X . Ultraweakly closed abelian A are precisely described in example 1 and we extend the usual duality $\langle L^1, L^\infty \rangle$ to this situation. Theorem 5.3 is of special importance.

Chapter 6 is an illustration of the technicalities and may be omitted upon first reading. Spaces A , in direct connexion with von Neumann algebra theory, are those for which A_i^{-1} is central (for a von Neumann algebra, one has

$A_i = Id$) and are easily handled. In particular, all classical results of C^* -theory concerning derivations are put in this context.

Chapter 7 is concerned with the notion of a representation. A representation π of A is a linear map from A into the space $B(\mathcal{D}_\pi, \mathcal{D}_\pi)$ of continuous sesquilinear forms on another domain $\mathcal{D}_\pi$ dense in some Hilbert space H_π : the restriction of π to the involutive algebra A_{id} - i.e., the bounded part of A - is a representation of this algebra in the Hilbert space H_π and π acts on the $\star$ -algebra $\mathcal{P}$ generated

by all A_i, A_i^{-1} as a representation from P into an involutive algebra of operators sending $\mathcal{D}_\pi$ into $\mathcal{D}_\pi$; one has $\mathcal{D}_\pi = \cap_{j \geq 0} \pi(A_j^{-1})H_\pi, \pi(A_i^{-1})^{-1} = \frac{\pi(A_i)}{\pi(A_i)}$ and $\pi(A_i)^\varepsilon \mathcal{D}_\pi = \mathcal{D}_\pi$

for $\varepsilon = \pm 1$ and all $i \geq 0$; the image of $\beta = BA_i \times A_i \in A$ where $B \in A_{id}$ is $\pi(\beta) = \pi(B)\pi(A_i) \times \pi(A_i) \in B(\mathcal{D}_\pi, \mathcal{D}_\pi)$. This definition appears both supple and stable under known operations such as subrepresentations, direct sums, tensorproducts, etc.

The case P abelian is essential (note the analogy with maximal abelian subalgebra of semi-simple complex Lie algebras and with Cartan subalgebra iff characters on P are normal); in particular, it is shown that a positive linear form f on A induces a G.N.S. representation π_f of A with the following properties:

1°/ the restriction π_f to A_{id} is the usual G.N.S. representation - with cyclic vector ζ_f in Hilbert space H_f - associated to the couple (A_{id}, f) ;

2°/ One has $\zeta_f \in \cap_{j \geq 0} \pi(A_j^{-1})(H_f) \equiv \mathcal{D}_f$ and $f(\beta) = (\pi(\beta)\zeta_f, \zeta_f)$

for all $\beta \in A$.

For applications, it seems sufficient to deal with A ultraweakly closed and $f \geq 0$ normal on A . Note that the map

$$T \in A \cap L(\mathcal{D}) \mapsto j(T) = \pi(T)\pi_f \in \mathcal{D}_f$$

extends continuously from A into $\mathcal{D}_f'$, i.e., for $\beta \in A$, $j(\beta)$ is a distribution on $\mathcal{D}_f$.

We include, as a supplement to Chapter 7, a paragraph on the second dual of A objects [10]; the results are natural generalizations of W^* -theory. In particular, the second dual of a space A , with condition II and cofinal abelian sequence A_i , is an ultraweakly closed space with condition II and a cofinal abelian sequence.

Chapter 8 states all properties in direct connexion with the strong dual A^ρ of A . For a space A with condition II, we introduce the convex set S of linear forms $f \geq 0$ on A , such that $f(1) = 1$. This set S imbeds into the compact set K - with $\sigma(A'_{id}, A_{id})$ topology - of linear forms $f \geq 0$ on A_{id} with $f(1) = 1$ and the topology to be chosen on S appears to be the topology induced by K . For $T \in A$, let $\varphi(T)$ be the function on S defined by $\varphi(T)f = f(T)$; each $\varphi(T)$ is a Borel function on the dense Borel set S in K , and $T \in A \mapsto \varphi(T)$ is a positive ρ -norm preserving map from $A_\mathbb{R}$ onto $\varphi(A)$ (proposition 8.5). We thus find (proposition 8.7) that, to each $f = f^* \in A^\rho$, it corresponds a real Radon measure μ on K , such that one has $|\mu|(K-S) = 0, \varphi(A) \subset L^1(S, d\mu)$ and

$f(T) = \int_S \varphi(T) d\mu$: equivalently, given an integer $i \geq 0$, one has the unique decomposition $f = f_1 - f_2$ on A with elements $f_1 \geq 0, f_2 \geq 0$ in A^0 and $\|f\|_{A_i} = \|f_1\|_{A_i} + \|f_2\|_{A_i}$ (for f in the predual P_A , one has $f_1, f_2 \in P_A$). This description is explicit for $B(\mathcal{D}, \mathcal{D})$ and other descriptions of the predual are indicated.

Chapter 9 introduces a notion of G -invariance. For an ultraweakly closed space A with condition II and cofinal abelian sequence A_i , we consider continuous linear forms f on A which are invariant under inner automorphisms associated to an abelian von Neumann algebra P containing all A_i^{-1} : linear forms invariant under inner automorphisms from $P^C = P' \cap A_{id}$ are called G -traces (this notation is due to the fact that one may choose $A_i = 1$ for a von Neumann algebra). For a G -invariant f , one can show that the above-mentioned decomposition $f = f_1 - f_2$ with $\|f\|_{A_i} = \|f_1\|_{A_i} + \|f_2\|_{A_i}$ for

a given i must satisfy $\|f\|_{A_j} = \|f_1\|_{A_j} + \|f_2\|_{A_j}$ for all $j \geq 0$

(this fact was suggested by H. Araki) and that the modular one-parameter group σ_t^f of (A_{id}, f) induces a one-parameter group of isomorphisms of A . Extremal positive G -invariant linear forms on A correspond to characters of the abelian ultraweakly closed $*$ -algebra $P = \bigcup_{i \geq 0} P A_i \times A_i$ and a precise

description of G -invariant states and G -traces is obtained when characters on A are normal.

Chapter 10 treats commutation theory as suggested by example 3. Due to the important role of $B(\mathcal{D}, \mathcal{D})$ - the analogue of the Banach algebra $L(H)$ in W^* -theory - one is tempted to calculate the commutant in $B(\mathcal{D}, \mathcal{D})$, as was done in [1]. Let us take an ultraweakly closed space A with condition II, and let M be the von Neumann algebra $M = A_{id}$. The striking fact is that a commutant for A does not necessarily lie in $B(\mathcal{D}, \mathcal{D})$, but has to be found in some $B(\mathcal{D}_1, \mathcal{D}_1)$, where $\mathcal{D}_1$ is a chosen dense domain 'affiliated' to the von Neumann algebra M' . The commutant $(A', \mathcal{D}_1)$ so obtained is an ultraweakly closed space with condition II' in $B(\mathcal{D}_1, \mathcal{D}_1)$

(this last ultraweak topology refers to $\mathcal{D}_1 \hat{\otimes} \mathcal{D}_1$) and the double commutant property follows. It is shown that commutation between $A \equiv (A, \mathcal{D})$ and $A' \equiv (A', \mathcal{D}_1)$ is ensured by two $*$ -algebras contained in A and A' , respectively, and dense in these spaces (theorems 10.1 and 10.3). It is also proved that $\mathcal{D} \cap \mathcal{D}_1$ must contain an invariant linear set dense in each of the Fréchet spaces $\mathcal{D}$ and $\mathcal{D}_1$ which suffices for all commutations properties (proposition 3-13): this is the analogue of C_c^∞ in example 3.

We point out that a normal, faithful linear form $f \geq 0$ on A determines a privileged commutant for $\pi_f(A)$, and that this commutant is anti-isomorphic to $\pi_f(A)$. Finally, the commutant of a given $*$ -algebra A_0 with condition II is identical to the commutant of its ultraweak closure A .

Chapter 11 introduces and studies strong and ultrastrong (also called σ -strong) topologies; these I have avoided previously in order to improve the readability of the book. These topologies behave as one expects and are the natural context for a generalized version of Kaplansky's density theorem.

VOCABULARY

Let A be a linear operator, not necessarily bounded, defined on a dense linear subset $\mathcal{D}$ of a Hilbert space H . The closure of A will be denoted by $\bar{A}$, the adjoint of A by A^+ , and when $\text{Dom } A^+$ contains $\mathcal{D}$, the restriction of A^+ to $\mathcal{D}$ by A^* . We denote the set of integers by $\mathbb{N}$.

A set A of linear operators will be called a $*$ -algebra on a domain $\mathcal{D}$ if the following conditions are satisfied:

- 1°/ A is symmetric, i.e., $\text{Dom } A^+ \supset \mathcal{D}$ and $A^* \in A$ for any $A \in A$.
- 2°/ $\mathcal{D}$ is invariant, i.e., $A\mathcal{D} \subset \mathcal{D}$ for any $A \in A$.
- 3°/ A is an algebra, i.e., $A + B \in A$, $AB \in A$, $cA \in A$ for any $A, B \in A$, and any complex number c .
- 4°/ A is unital, i.e., $1 \in A$.

An operator $A \in A$ is positive (written $A \in A^+$, or $A \geq 0$), if $(Ax, x) \geq 0$ for all $x \in \mathcal{D}$. The positive cone A^+ is generating in A and defines the order structure $\leq$ of A . For every symmetric operator T in A , there exists a positive element A in A such that $-A \leq T \leq A$: this will be written simply as $|T| \leq A$. Moreover, we always assume that A is countably dominated, i.e., there exists a sequence A_n in A^+ , which is cofinal in A^+ .

The domain $\mathcal{D}$ of any $*$ -algebra A will be equipped with the topology given by semi-norms $x \in \mathcal{D} \rightarrow \|Ax\|$, with A varying in A . We note that A is countably dominated (by the sequence A_n) iff $\mathcal{D}$ is metric under this topology; in which case, semi-norms $x \in \mathcal{D} \rightarrow \|A_n x\|$ with $n \in \mathbb{N}$ suffice to define the topology of $\mathcal{D}$. The completion $\hat{\mathcal{D}}$ of $\mathcal{D}$ is $\hat{\mathcal{D}} = \bigcap_{A \in A} \text{Dom}(\bar{A})$ with

semi-norms $x \in \mathcal{D} \rightarrow \|\bar{A}x\|$, $A \in A$, and our $*$ -algebra A on the domain $\mathcal{D}$ can be considered as a $*$ -algebra A on the domain $\hat{\mathcal{D}}$. When $\mathcal{D} = \hat{\mathcal{D}}$, we say that $\mathcal{D}$ is the natural domain of A . Before defining the topology ρ on A [11], which rests on the order structure of A , we note that $(Ax, x) \geq 0$ for all $x \in \mathcal{D}$ is equivalent to $(\bar{A}x, x) \geq 0$ for all $x \in \hat{\mathcal{D}}$ (for some A in A), and it follows that we may, and do, assume, without loss of generality, the equality $\mathcal{D} = \hat{\mathcal{D}}$.

Algebras handled in this paper will satisfy:

Condition I. There exists a cofinal sequence A_n in A^+ , such that $A_n \geq 1$, and $A_n^{-1} \in A$ for every n .

Condition I clearly imply $A_n \mathcal{D} = \mathcal{D}$ for every integer n ; however, one has to point out that it is not true, in general, that $A\mathcal{D} = \mathcal{D}$ for any $A \geq \text{Id}$, $A \in \mathcal{D}$. A weaker condition, often met in practice (in representations of enveloping algebras, for example) is

Condition I_0 . There exists a cofinal sequence A_n in A^+ such that $A_n \geq \text{Id}$ and $A_n \mathcal{D} = \mathcal{D}$ for every $n \in \mathbb{N}$.

When condition I_0 is satisfied, the algebra A_1 generated by A and all A_n^{-1} (restricted to $\mathcal{D}$) is a $*$ -algebra on $\mathcal{D}$ satisfying condition I and the commutant A' of A and A_1 (A' is calculated inside $B(\mathcal{D}, \mathcal{D})$) are the same. It is seen in [1], theorem 3 that A' is always a $*$ -algebra on the domain $\mathcal{D}$, however, the natural domain D of A' is, in general, larger than $\mathcal{D}$.

At this point, there are two possibilities - a priori - for a second commutant A'' for A :

1°/ The commutant of $(A', \mathcal{D})$ calculated inside $B(\mathcal{D}, \mathcal{D})$ leads to the σ -weak closure A_1^σ - relative to $\mathcal{D} \hat{\otimes} \mathcal{D}$ - of the $*$ -algebra A_1 . Note that A_1^σ is not, in general, a $*$ -algebra.

2°/ The commutant of (A', D) calculated inside $B(D, D)$ leads to an ultraweak closed $*$ -algebra - relatively to $D \hat{\otimes} D$ - with domain D . It often happens that this second commutant - always contained in A_1^σ - may consist of bounded operators only.

In fact, it is seen in Chapter 10 that commutation theory is more simple, and is a simple reality suggested by examples.

The complex conjugate $\mathcal{D}^-$ of $\mathcal{D} (\equiv \hat{\mathcal{D}})$ will be needed all through the book. As no confusion is possible in these notes, I have adopted a unified notation, i.e., $\mathcal{D} = \mathcal{D}^-$. I recall that the dual of the projective tensor product $\mathcal{D} \hat{\otimes} \mathcal{D}$ is the set $B(\mathcal{D}, \mathcal{D})$ of continuous sesquilinear forms on $\mathcal{D} \times \mathcal{D}$ (in the particular case when $\mathcal{D}$ is the Hilbert space H itself, $B(\mathcal{D}, \mathcal{D})$ coincide with the Banach algebra $L(H)$ of bounded linear operators acting in H). Let $\mathcal{B}$ be a linear subset of $B(\mathcal{D}, \mathcal{D})$; the topologies on $\mathcal{B}$ induced by $\sigma(B(\mathcal{D}, \mathcal{D}), \mathcal{D} \hat{\otimes} \mathcal{D})$ and $\sigma(B(\mathcal{D}, \mathcal{D}), \mathcal{D} \hat{\otimes} \mathcal{D})$ are called the weak and ultraweak (or σ -weak) topology of $\mathcal{B}$. Thus, weak topology is defined by semi-norms $\beta \in \mathcal{B} \rightarrow |\beta(x, y)|$, with x and y varying in $\mathcal{D}$, and σ -weak topology by semi-norms $\beta \in \mathcal{B} \rightarrow |\sum_{i=1}^{\infty} \beta(x_i, y_i)|$, where $x = (x_i)_{i \in \mathbb{N}}$ and $y = (y_i)_{i \in \mathbb{N}}$ vary in the set of σ -convergent sequences (i.e., $\sum_{i=1}^{\infty} \|Ax_i\|^2 < +\infty$ for every $A \in A$ [11])

Any operator T of the $*$ -algebra A is identified to the continuous sesquilinear form $(x, y) \in \mathcal{D} \times \mathcal{D} \rightarrow (Tx, y)$, so

that A is viewed as a vector subspace of $B(\mathcal{D}, \mathcal{D})$. In fact, the set $B^+(\mathcal{D}, \mathcal{D})$ of all β in $B(\mathcal{D}, \mathcal{D})$, such that $\beta(x, x) \geq 0$ for all x of $\mathcal{D}$, induces the order structure of A , and defines the order structure $\leq$ in $B(\mathcal{D}, \mathcal{D})$: an important point is that any cofinal subset of A^+ is automatically cofinal in $B^+(\mathcal{D}, \mathcal{D})$. The map $\beta \in B(\mathcal{D}, \mathcal{D}) \mapsto \beta^* \in B(\mathcal{D}, \mathcal{D})$, where, by definition,

$\beta^*(x, y) = \overline{\beta(y, x)}$ for $x, y \in \mathcal{D}$ is an involution on $B(\mathcal{D}, \mathcal{D})$, and extends the involution of A . Finally, for $\beta = \beta^*$ in B , and $A \geq 0$ in B , $|\beta| \leq A$ means $-A \leq \beta \leq A$.

For any subspace B of $B(\mathcal{D}, \mathcal{D})$ with $B = B_R \oplus iB_R$, where B_R is the set of all hermitian β in B (i.e., $\beta = \beta^*$), B containing a cofinal subset of the positive cone A^+ of the given $*$ -algebra A , we define the ρ -topology as follows. For $A \in B^+$, the set of all $T \in B_R$, such that $|T| \leq \lambda A$ for some $\lambda \in \mathbb{R}$, is denoted by B_A and the infimum of λ is denoted by $\rho_A(T)$ or $\|T\|_A$. Clearly, $B_R = \bigcup_{A \geq 0} B_A$ and the locally convex inductive limit topology for the system of normed spaces (B_A, ρ_A) is the ρ -topology of B_R ; the topology ρ of B is the direct topological sum $B_R \oplus iB_R$. Since A is countably dominated, we may choose a monotone increasing sequence A_n in A^+ (or in B^+) such that $B_R = \bigcup_n B_{A_n}$, and (B_R, ρ) coincide with the inductive limit of the sequence of normed space $(B_{A_n}, \|\cdot\|_{A_n})$.

We often identify (B_R, ρ) with (B, ρ) in our notations. Let us recall that (A, ρ) is a separated $\mathcal{DF}$ -space, that the strong dual of (A, ρ) is always a Fréchet space, and that any positive (resp. bijective positive) linear map from B into a space of the same type decreases (resp. preserve) the ρ -norms, hence is continuous (resp. bicontinuous).

Of course, one can consider any $*$ -algebra A , with domain $\mathcal{D}$, as a vector subspace of the set $B(\mathcal{D}, H)$ of continuous sesquilinear forms on the product $\mathcal{D} \times H$. Clearly, $B(\mathcal{D}, H)$ can be identified with the set of linear operators continuous from the Fréchet space $\mathcal{D}$ into the Hilbert space H , and $B(\mathcal{D}, H)$ is the dual of the projective tensor product $\mathcal{D} \hat{\otimes} H$; thus, in this context, it is possible to introduce weak topology, or σ -weak topology, with reference to $\mathcal{D} \hat{\otimes} H$ or $\mathcal{D} \hat{\otimes} H$. This embedding of A into $B(\mathcal{D}, H)$ leads to a topology, called λ , which happens to be equal to ρ for a particular $*$ -algebras (see [11] and paragraph 6). For any $*$ -algebra A with domain $\mathcal{D}$ (or more generally, for a subspace of $B(\mathcal{D}, H)$), we introduce, for $T \in A$, and $0 \neq A \in A$, the quantity

$$\lambda_A(T) = \sup(\|Tx\| / \|Ax\|; x \in \mathcal{D})$$

with $a/0 = +\infty$ for $a > 0$, and the set A^A of all T in A being such that $\lambda_A(T)$ is finite. Each A^A is a linear space with

natural norm λ_A , and λ -topology is, by definition, the inductive limit topology of the system of normed spaces (A^A, λ_A) , $A \in A$. This topology is not related to the order structure of A .
Subspaces B of $B(\mathcal{D}, \mathcal{D})$, of special interest, satisfy condition II.

Condition II. B is a linear subset of $B(\mathcal{D}, \mathcal{D})$, stable under involution and containing a cofinal subset A_n of $B^+(\mathcal{D}, \mathcal{D})$, with $A_n \geq 1$, $A_n(\mathcal{D}) = \mathcal{D}$ for every n , satisfying the following:

1°/ B_{id} is an involutive algebra of bounded operators containing 1.

2°/ $\beta \in B$ implies $A \beta A_n \in B$ and $A_n^{-1} \beta A_n^{-1} \in B$, where $(C \beta D)(x, y) = \beta(Dx, C^*y)$.

In general, elements of B_{id} do not send the domain $\mathcal{D}$ into itself. For any $B \in B_{id}$ and any integer n , the continuous sesquilinear form $\beta : (x, y) \in \mathcal{D} \times \mathcal{D} \rightarrow (B A_n x, A_n y)$ belongs to B , by 2°/, and will often be denoted by $BA_n \times A_n = \beta$.

The space $B(\mathcal{D}, \mathcal{D})$ satisfies condition II, and $B(\mathcal{D}, \mathcal{D})_{id}$ is the usual Banach algebra $L(H)$, H being the Hilbert space completion of $\mathcal{D}$. The meaning of condition II comes from the following situation: for $*$ -algebra A satisfying condition I, A_{id} is an involutive algebra of operators sending $\mathcal{D}$ into $\mathcal{D}$, the maps introduced in condition II 2°/ being the usual

multiplication $T \in A \rightarrow A_n T A_n \in A$ and $T \in A \rightarrow$

$A_n^{-1} T A_n^{-1} \in A$, and the σ -weak closure A^σ of A verify condition II, the set $(A^\sigma)_{id}$ being exactly the von Neumann algebra generated by A_{id} [1].

For a $*$ -algebra A , conditions I and II are identical.

Finally, we will use also notation $C \beta D = \beta(D^* \cdot, C \cdot)$.

By a space A with condition II and natural domain $\mathcal{D}$, we mean that $\mathcal{D}$ is the natural domain of the $*$ -algebra generated by all A_n, A_n^{-1} .