Mathematical Material for Chapter III

Grains on the Chessboard

Chapter III deals with the permanent general goals that are pursued by Wiskobas.

The theme "Grains on the Chess-board" — designed for the sixth grade — serves to illustrate the so-called one-dimensional goals.

It is important for a good understanding of the contents of the general goals that the reader not only deals with the mathematical problem but is also aware of the ways in which pupils solve the given problem, and which are collated in the commentary at the end of this section on the mathematical material.

Designed by USMES-project (M. Lazarus). Revised version.

GRAINS ON THE CHESSBOARD

1. *Grains of Wheat*

According to the legend, the inventor of chess brought his new game to his king. The monarch was delighted by it and offered the inventor a reward of his own choice.

. . . "Your Majesty, this is my wish: give me my reward in grain, measured in the following way: one grain on the first square, two on the second, four on the third, eight on the fourth, and so on for each square."

The king was astonished: "Is that all you want? Only grain? Very well, so be it. Fetch a full sack of grain."

A strong servant left to do his master's bidding.

▷ Will one sack of grain be enough?

2. *A Faster Count*

One hour later . . . . Several men were busily counting the grains. The king was becoming slightly uneasy.

"Haven't you finished yet?" he asked.

"Oh no, my lord, we've only just reached the 14th square."

"Only the 14th? Why is it taking so long?"

"Sire, for the 14th square alone we need thousands of grains: it will be some time before we have finished counting them."

"This could take all night!", the king sighed.

No one heard the inventor mutter under his breath: "I'm afraid it will take a lot longer than that."

"Can't you count less precisely and more quickly?", the king begged.

The inventor intervened and said "I have an idea that will speed things up." . . .

How long will it take to count the grains for the 14th square?

▷ "That could take all night", the king sighed.

How long will it really take? Can you think of a quick way of making an estimate of the number needed?

3. *More Sacks of Grain*

The inventor explained to the king how the counting could be speeded up.

. . . "A good idea," the king said, "fetch a scoop and a pair of scales." . . .

Some time later .... A number of servants were still scooping and weighing.

The king was pacing up and down. One of the servants approached the king to inform him that the sack of grain was empty.

"What, do we need more?" he exclaimed.

The servant answered, "Yes, Sire, much more."

"Very well, fetch a new sack, and just to be sure, bring an extra sack as well."

Two servants were sent out to do so.

▷ Which square do you think the servants had reached when the first sack of grain became empty? Will two extra sacks of grain be enough? Explain your answer.

4. *A Kingdom of Grain*

Later still .... Sacks of grain filled up most of the throne room when it was reported that the royal stores were empty. Now what?

The chess master said: "Majesty, give me the rest later, but before I go I would like to know how much you still owe me."

"Very well", said the king, "call for the royal treasurer."

Shortly after he had received the king's order, the treasurer said "Sire, I have bad news for you, there is not nearly enough grain in your entire kingdom to pay for this."

"But that's impossible", the king declared, "it is such a simple request: first one grain, then two, then four, then eight..."

"Certainly", the treasurer replied: "At the beginning the amounts were small, but they grew rapidly, the 11th square has 1024 grains, let us say 1000 to make it easier, that is one scoop of grain. The 1000 scoops for the 21st square fill one sack. From this point on the number of sacks are doubled. The 31st square requires 1000 sacks... a barnful!"

▷ In the story it is said that the sacks of the grain filled up most of the throne room. How many squares had been covered at that point?

▷ Is the treasurer's calculation correct? Follow his argument from the 41st to the 64th square in such a way that it will express the amount for the last square in comprehensible terms.

5. *The Calculation*

The royal treasurer went on. He spoke of thousands of rooms of grain and of figures with more than ten digits and showed how to determine the number of grains for the last square. The king showed the results to the inventor.

"Sire," the inventor replied. "to be honest, I am not completely satisfied. ... This amount gives me only the amount of grain for the last square, while you promised me the total of all of the squares together."

The king recalled the treasurer and demanded an explanation. He had not known of the king's promise.

"Nevertheless, I want to know the total, and quickly at that, I want to go to bed", the king ordered.

The treasurer sat down on one of the empty sacks and started his calculation.

The inventor watched over his shoulder and the king, who was bored by now, let some grain slip through his fingers.

▷ How many digits are there in the number that expresses the number of grains in the last square (working with rounded-off numbers)?

▷ Calculate the exact number of grains on the last square. (A pocket calculator may be used, even though it is not equipped to give the final result).

▷ Calculate the exact total number of grains (Give this assignment only if the addition rule, from the previous sections, has already been discovered. See Section 3 of the Comments for a description of the "addition rule").

6. *The Dream*

The treasurer, who was a mathematician, discovered that the total number of grains on the first four squares was one less than the number of grains on the fifth square; the total number on the first five squares was one less than the number of grains on the sixth square, etc. He showed the king that in this way the total number of grains could be determined quickly once the amount for the last square was known.

This was too much for the king, but one thing was very clear to him: he had been caught, even if it was in an honest fashion.

The inventor had slipped away unnoticed since he realised that the king's mood was none too good. The king looked around for him, but not seeing him anywhere, muttered something to the treasurer and turned to leave the throne room but not before ordering: "Tell the farmers to grow more grain."

He went to his royal bedroom and once asleep he had the most extraordinary dream.

▷ Give your version of the king's dream.

COMMENTS

1. *Grains of Wheat*

The class was allowed only a short time to think. Therefore the answers were wild guesses. Were the children aware of the fact that some important information is missing?

Their estimates varied greatly: from 6000 to a few million grains. A few pupils doubted whether one sack of grain would be sufficient. The different groups each explained their estimates. The feeling in general was to question: who is right?

2. *A Faster Count*

The class found two ways to determine how long it will take to count: by agreement (1 count per second, therefore . . .) and by measuring (50 grains per minute, so. . .).

General conclusion: count on indefinitely.

The teacher showed the class an easier method: approx. 1 minute for the 7th square (64 grains); 2 minutes for the 8th square; 4 for the 9th; 8 for the 10th; 16 for the 11th; 32 for the 12th; 64 for the 13th (about 1 hour); 2 hours for the 14th square, and so on.

The class found two ways to conduct an indirect count: scooping and weighing.

One thousand grains (the rounded-off number for the 11th square) weigh approx. 30 grams and fill a measuring glass of 50 cm^3.

In conclusion, the teacher asked: "how much do the grains on the 14th square weigh? On the 15th? On the 16th square? Which square yields 2 dl and which will yield a little less than 1 dl? Will one sack of grain be enough?"

By now most of the groups doubted whether one sack of grain would be sufficient.

3. *More Sacks of Grain*

The contents of one sack of grain was estimated, then measured and finally found to be $\frac{1}{2}$ hl.

Agreement: the amount on the 11th square ($\frac{1}{2}$ dl.) is one scoop.

Question: how many scoops to one sack? Some guessed, others calculated with cubic measures. The 11th square equals one scoop, therefore the 21st square equals 1024 — say 1000 scoops, or 1 sack.

Up to this point the teacher had held a tight rein on the amount of material used.

This was now the time for class discussion. Will they discover that the number of grains on a certain square is one more than the total number of grains on the previous squares? (In this context "one more than" can be considered as "equal to"). If not, it can be left undiscovered and a rough addition can be made.

Conclusion: the sack will be empty when the 21st square is reached. The teacher can illustrate the answer using fractions: calculating backwards from the 21st square to show that $1/2 + 1/4 + 1/8 + 1/16 \dots$ is approximately 1.

The second question can now be answered: the class came to the conclusion that the second sack will not be enough to fill the 23rd square. By the end of the second lesson the pupils had noticed the "unimaginable growth", which will be made "imaginable" in the next section.

4. *A Kingdom of Grain*

The children chose their own dimensions for the throne room, calculated the volume in m^3, transformed this into an estimate of the number of sacks and tried to estimate which square will be reached. Some pupils substituted 10hl for 1 m^3, that is 20 sacks of $\frac{1}{2}$ hl each, which is the treasurer's measure. A few decided to work with barns full of grain, just as suggested later on in the story.

Not only did the solution strategies vary — the estimates for the size of the throne room were even more divergent.

Both of these points were then discussed briefly.

The capacity of the barn or the classroom — 1000 sacks — offered a good basis. Again the addition rule became important. It is possible that the relationship between the number of grains on one square and the total of the previous squares will then be found.

In any case retracing the steps like the teacher did for the sacks of grain will be helpful. Calculating the number of barns may also he useful.

Conclusion: Depending on the size of the throne room, the entire supply of grain will be used up by the time the 40th square is reached. To make this imaginable: throne room (41st square), sky-scraper (51st square), city (61st square).

The amount for the last square: a cube-shaped chest with an edge of 8 km. The Hague in the shape of a cube!

By the end of the third lesson the problem had almost been solved. A few surprising finishing touches remained.

5. *The Calculation*

So far the pupils had discovered that a jump across ten squares yields a thousand fold increase in grains. This rough estimate can be used to determine the number of grains on the last square: 11th square 1000 grains; 21st square 1,000,000; 31st square 1,000,000,000; and the 61st square "a one with 18 zeros" (a number with 19 digits).

This, multiplied by eight for the 64th square, results in a number of 19 or 20 digits — the rounding off procedure was a source of uncertainty. Now the exact number for the 64th square could be found so that a check could be made. The way in which this was to be done was discussed. Repeatedly the teacher asked the class: "Who can think of a better way?"

The pocket calculator cannot be used straightaway. It cannot express a 19 digit number.

This assignment can be given to pupils who have already learned to work in other number systems: Write the number of grains of the 64th square in the binary system.

The answer is quite a revelation: a one with 63 zeros. The same can be done in base four and base eight. Pleasing number patterns emerge. By the end of lesson four the problem had been completely solved. The theme was rounded-off by the last part of the story.

6. *The King's Dream*

A few facts: the total number of grains is 18,446,744,073,709,551,615.

World production of grain for 1976 was 1 billion metric tons.

Most of the lesson was taken up by writing the story of the king's dream. The teacher also made his own version.

One short example: The king dreams of a huge chess board set out on the black and white tiles of the courtyard. The servants count out grain according to the inventor's method.

One sack fulfils the requirements for the 21st square; two come on the 22nd square; four sacks for the 23rd square. The king climbs up the steps of the tower rising above the next square. One thousand sacks are needed for the 31st square. From his position in this tower, at 1 km up, the king can survey most of his kingdom. At the 42nd square the king finds himself 1000 km above the ground.

And still the servants keep piling up grain.

Soon the king passes the moon, the planets, the sun and finally disappears into space filled with glistening grains.

The tower on the last square begins to sway and the king plunges into the dark depths beneath him. He wakes up with a scream of terror.

The teacher used his version to illustrate a graph showing exponential growth. Furthermore, he showed that such growth could only be portrayed to a very limited extent.