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ADRIAN TREFFERS

*Mathematics Education Research Group (OW & OC),
State University of Utrecht, The Netherlands*

THREE DIMENSIONS

*A Model of Goal and Theory Description in
Mathematics Instruction — The Wiskobas Project*

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*Dedicated to Hannie
Hanneke
Ine*

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PREFACE

In Dutch “WISKOBAS” stands for a particular kind of mathematics in the elementary school (ages 6–12). In turn Wiskobas was one of the departments in the IOWO, the Institute for the Development of Mathematics Education. This institute was concerned with the development of material for mathematics education as well as the related research on the possibility of change from the then existing arithmetic instruction to the future mathematics education. The present publication *Three Dimensions* has three aims: to give a picture of the goals Wiskobas set for future mathematics education, at the same time to show how such goals can be described, and to show the theoretical framework of the Wiskobas curriculum.

The problem at hand is not at all simple. What is more, Wiskobas’ ideas about mathematics education cannot literally be translated into strings of words. So how can we face the accusation that our objectives are unattainable and the goal itself irrational? In order to avoid this vagueness as much as possible and for the sake of clarity, this book makes continuous use of illustrations of mathematics education. In these examples both the subject-matter and the methods of description of the goals are illustrated as explicitly as possible, while at the same time creating the opportunity to read between the lines. The reader is urged to follow carefully the mathematical material at the start of each chapter. This advice applies both to the more general education oriented, and to the more mathematical/didactical reader.

In the completion of this task I received assistance from Jelle Sixma, Hans Freudenthal and the former Wiskobas team of the IOWO (now the OW & OC at the State University of Utrecht). A special word of thanks is due to Rob de Jong, Sylvia Pieters, Betty Dekker and Els Feijs, and to Hetty and Guus Vonk, Hans Freudenthal, Alan Bishop and the late Arthur Morley for their help in translating the Dutch version into English.

Chapters I–V are translated from the original version. Chapter VI is new and was written in 1983. It contains an example of the goal description of a course, as opposed to the examples of goal descriptions in the other chapters which have a bearing on themes. Chapter VII was appended in 1986 with the idea that the three-dimensional goal description propagated here only becomes transparent when a clear view is provided for the framework of instruction theory. In our case, this concerns mathematics education as realised by the Wiskobas project. This theoretical framework is also described in three dimensions, which

explains the title of this study, “A model of goal and theory description . . .”. The goal description is the principal part of this study — in this case the primary school — and the theory description is dealt with extensively in the Appendix (Chapter VII) in order, as mentioned above, to clarify the goal description.

GUIDE TO THE MATHEMATICAL MATERIAL

In order to deal with the contents and methods describing the goals of mathematics education, each chapter of the present book starts with a piece of mathematical material for primary school use.

The function of the mathematical material is varied. Generally speaking, however, the reader is given the opportunity to acquaint himself with the kinds of experience of mathematics education in the elementary school that were propagated by Wiskobas. The reader may use it by reading and working out the problems presented; and at the same time a concrete foundation for considering the problem of goal description will be laid.

On the other hand the examples of mathematical material enable the author to support his views concerning goal descriptions, in respect to both form and content. Besides the general aims of providing acquaintance, explanation and support, each particular piece of mathematical material in each chapter has its own specific function, which is indicated immediately preceding the material.