Complex Analytic Sets

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Complex Analytic Sets

by

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Translated from the Russian by R. A. M. Hoksbergen

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SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir,
je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be
able to do something with it.

O. Heavyside

One service mathematics has rendered the
human race. It has put common sense back
where it belongs, on the topmost shelf next
to the dusty canister labelled 'discarded non-
sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the *raison d'être* of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

"Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the

extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the nonlinear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p -adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five subseries: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had on the development of another.

This volume in the series is about analytic sets in the sense of the theory of functions of several complex variables (or analytic geometry as that phrase is used in the modern sense of the word in analogy with algebraic geometry). That is, in this book an analytic set is a set in a complex analytic manifold that is locally given by the zeros of a finite number of holomorphic functions (and it is not a complete separable metric continuous image of the set of irrational numbers or one of the various generalizations of that idea).

The concept has been around for a while and several books, mostly called "several complex variables" or "analytic varieties" devote space to the topic, especially in the form of the local theory and, much related to that, the theory of singularities. As a separate subject within several complex variables, important for a wide range of problems involving the analysis and construction of holomorphic mappings, and stressing - though the local theory remains important - more global and geometric aspects (such as analytic covers, metrical properties) and boundary properties, the field is a rather young one.

It is a pleasure to welcome this systematic and thorough treatment of the topic by a very well known author in the field in this series.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Bussum, June 1989

Michiel Hazewinkel

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Preface

The theory of complex analytic sets is part of the modern geometrical theory of functions of several complex variables. A wide circle of problems in multidimensional complex analysis, related to holomorphic functions and maps, can be reformulated in terms of analytic sets. In these reformulations additional phenomena may emerge, while for the proofs new methods are necessary. (As an example we can mention the boundary properties of conformal maps of domains in the plane, which may be studied by means of the boundary properties of the graphs of such maps, cf. article [184] by the author.)

The theory of complex analytic sets is a relatively young branch of complex analysis. Basically, it was developed to fulfill the need of the theory of functions of several complex variables, but for a long time its development was, so to speak, within the framework of algebraic geometry - by analogy with algebraic sets. And although at present the basic methods of the theory of analytic sets are related with analysis and geometry, the foundations of the theory are expounded in the purely algebraic language of ideals in commutative algebras.

In the present book I have tried to eliminate this noncorrespondence and to give a geometric exposition of the foundations of the theory of complex analytic sets, using only classical complex analysis and a minimum of algebra (well-known properties of polynomials of one variable). Moreover, it must of course be taken into consideration that algebraic geometry is one of the most important domains of application of the theory of analytic sets, and hence a lot of attention is given in the present book to algebraic sets.

The known monographs on complex analytic sets (Abhyankar [1], Hervé [203], Whitney [150], Gunning and Rossi [35], etc.) are, basically, concerned with the so-called local theory. This direction is related to the theory of singularities of differentiable maps, and is gradually absorbed by it. In the present book, local problems also take an important place, but the main goal was the exposition of global properties of analytic sets as the geometric analog of holomorphic functions and holomorphic maps.

The book consists of four chapters and an appendix. Chapter 1 contains the base of the theory. As already noted, the present exposition is distinguished by the extensive use of geometric and analytic methods. Theorems on removing singularities of holomorphic functions of several variables and simple geometric theorems on proper projections of sets in $\mathbb{C}^n$ lie at their basis. The notion of analytic cover, which was developed in detail in the book by Gunning and Rossi [35], plays an important part in these problems. Canonical defining functions of finite tuples of points in $\mathbb{C}^n$, analogs of the standard polynomials $\prod (z - a_j)$ on the plane, also proved to be a convenient tool. These functions were systematically used in Whitney's book [150].

Chapter 2 is concerned with infinitesimal properties of analytic sets and intersection theory. The paragraphs on tangent cones are written under extensive influence of Whitney's work [148], [149], [150], the exposition of the intersection theory of holomorphic chains is close to Draper's paper [53], but gives proofs of more geometrical nature.

In Chapter 3 we study metrical properties and other problems related to the integration of differential forms around analytic sets. These problems are closely related to multidimensional Nevanlinna theory, which was intensively developed in the last years (cf. [42], [186]).

Chapter 4 is, perhaps, closest of all to function theory. In it we study parallels with theorems on analytic continuation and on boundary properties of holomorphic functions of several variables.

In the Appendix we have collected some necessary information from multidimensional complex analysis; the results, which are included in the well-known handbooks and monographs [32], [35], [175], [185], are moreover expounded very quickly. Since Hausdorff measures play an important part in the exposition given, at the end of the Appendix proofs of the necessary properties of these measures are given.

Some remarks on what I would have liked to include also in the book. In the local problems - Milnor's theory of isolated singularities [87] and its generalizations. In Chapter 3 a paragraph on volumes of tubes over analytic sets and their relations with Chern classes and curvatures, in the spirit of Griffiths' article [41], would have been appropriate. In Chapter 4 I would have liked to include additional results on boundary properties of analytic sets, in particular those from

[184]. Moreover, it was the intention to give an exposition of the properties of holomorphic, weakly holomorphic, and other functions on analytic sets. Unfortunately, shortage of space and time did not allow me to include this material.

In order to give an exposition of the foundations of the theory by a new method a number of technical results had to be proved, using a minimum of prerequisites. All these results are generally known and have been published somewhere or belong to mathematical folklore; it is difficult in these cases to indicate sources. This also pertains to a lot of material of the subsequent chapters: absence of authorship indication simply means that I could not succeed in exactly establishing the author of a given result, and by no means indicates that this result belongs to the author of the present book. For reference purposes an extensive bibliography is given at the end of the book; it includes classical works and works of recent years.

References within the book are divided into paragraphs and sections. Numbering of paragraphs within the main text (Chapters 1-4) is compound, paragraphs are divided into sections, a reference like 'p.11.3' means section 3 of paragraph 11. The Appendix has its own numbering of paragraphs, e.g. 'A5.4' means section 4 of paragraph 5 of the Appendix. A reference like 'Theorem 13.5' (Lemma, Proposition) means the theorem in p.13.5 (if there is only one); within sections there is no such numbering and, e.g., 'Theorem 2' means Theorem 2 of the present section.

The sign ■ indicates the begin and end of a proof.

Finally, for their critical remarks and constant stimulating support I consider it my pleasant duty to thank my colleagues and friends, especially S.I. Pinchuk, A. Sadullaev, and B.V. Shabat.

E.M. Chirka

List of notations

$\mathbb{C}$; the complex plane, the field of complex numbers;

$\mathbb{R}$; the real line, the field of real numbers;

$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$;

$\mathbb{Z}$; the integers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$;

$\mathbb{N}$; the natural numbers;

$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times);

$(\mathbb{C}^n)^*$; the space of $\mathbb{C}$ -linear functions on $\mathbb{C}^n$ (the dual of $\mathbb{C}^n$);

$(\mathbb{C}^n)^*_{\mathbb{R}}$; the space of $\mathbb{R}$ -linear complex valued functions on $\mathbb{C}^n$;

$\mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\}$;

$\mathbb{P}_n$; complex n -dimensional projective space (see A3.2);

$G(k, m)$; the Grassmannian of complex k -dimensional subspaces in $\mathbb{C}^n$ (see A3.4);

$(z, w) = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$; the Hermitian scalar product in $\mathbb{C}^n$;

$|z| = (z, z)^{1/2} \geq 0$;

$\langle z, w \rangle = z_1 w_1 + \cdots + z_n w_n$;

$\mathbb{C}^n \approx \mathbb{R}^{2n}$; $(z_k = x_k + iy_k)$, $(z_1, \dots, z_n) \leftrightarrow (x_1, y_1, \dots, x_n, y_n)$, where the isomorphism $\mathbb{R}^n \subset \mathbb{C}^n$ is the natural inclusion $x = (x_1, \dots, x_n) \in \mathbb{C}^n$;

$B(a, r)$; the ball in $\mathbb{C}^n$ or $\mathbb{R}^n$ with center a and radius r ;

$U(n)$; the group of unitary transformations of $\mathbb{C}^n$ (see A2.1);

$[z] \triangleq [z'_0, \dots, z'_n]$; the point of $\mathbb{P}_n$ with homogeneous coordinates $(z_0, \dots, z_n)$;

$\Pi: \mathbb{C}_*^{n+1} \rightarrow \mathbb{P}_n$; the canonical projection $z \rightarrow [z]$;

$L^\perp$; the subspace of $\mathbb{C}^n$ orthogonal to $L \subset \mathbb{C}^n$, or the complex plane in $\mathbb{P}_n$ orthogonal to the plane $L \subset \mathbb{P}_n$ (see A3.3);

Notations with multi-indices $I = (i_1, \dots, i_p)$, $i_k \in \mathbb{Z}_+$.

$\#I$; the number of elements (the length) of I ;

$|I| = i_1 + \cdots + i_p \quad (p = \#I)$;

$z_I = (z_{i_1}, \dots, z_{i_p})$;

$\mathbb{C}_I = \mathbb{C}_{i_1 \dots i_p} = \mathbb{C}_{z_I}$; the coordinate plane of the variable z_I ;

π_I ; the orthogonal projection of $\mathbb{C}^n$ onto $\mathbb{C}_I$;

$z^I = z_1^{i_1} \dots z_n^{i_n}$ ($\#I = n$);

$$\frac{\partial |I|}{\partial z^I} = \frac{\partial |I|}{\partial z_1^{i_1} \dots \partial z_n^{i_n}};$$

I is ordered if $i_1 < \dots < i_p$;

$\sum'_{\#I=p}$; summation over the ordered multi-indices of length p ;

$\overline{E}$; the closure of a set E ;

$$\partial E = \overline{E} \setminus E;$$

$\#E$; the number of elements of a set E ;

$:=$ (or $=$); equality by definition;

δ_{jk} ; the Kronecker symbol ($= 1$ if $j = k$ and $= 0$ if $j \neq k$);

$E + c = \{x + c : x \in E\}$ in $\mathbb{C}^n$ or in $\mathbb{R}^n$;

$\wedge$; exterior product (of polyvectors, forms, linear spaces) (see A4.1);

dist; the distance;

det; the determinant of a matrix;

All manifolds in this book are assumed to be countably compact, i.e. to be representable as a countable union of compact subsets. The phrase “manifold M in Ω ” means manifold located in Ω , i.e. a submanifold of a neighborhood of it in Ω .

bM ; the boundary of a manifold M (with the induced orientation);

$T_a M$; the tangent space to a manifold M at a point a ; for manifolds in $\mathbb{R}^N$ it is usually regarded as a subspace of $\mathbb{R}^N \approx T_a \mathbb{R}^N$;

$T_a^c M = (T_a M) \cap i(T_a M) \subset T_a \Omega$; the complex tangent plane to M , i.e. the complex component of $T_a M$ in $T_a \Omega$ (if Ω is complex), see A2.4;

$\text{rank}_z f$; the rank of a differentiable map f at a point z (the complex rank if f is holomorphic);

Z_f ; the zero set of a function f ;

- $L^1_{\text{loc}}(U)$; the set of functions integrable on compact subsets of U (of locally integrable functions in U);
- $C^k(U)$; the set of k times continuously differentiable functions on an open subset U of a sufficiently smooth manifold;
- $\mathcal{D}(U)$; the set of differential forms of class C^∞ with compact support in U ;
- $\phi^{p,q}$; the component of a form ϕ of bidegree (p,q) on a complex manifold (the (p,q) -component of ϕ), see A4.2;
- vol_k ; the volume of subsets of a k -dimensional Riemannian manifold;
- $\mathcal{H}_\alpha$; the Hausdorff measure of order (or dimension) α (in other words, the α -measure), see A6.1;