The Homology of Banach and Topological Algebras

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The Homology of Banach and Topological Algebras

by

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Translated from the Russian by Alan West

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SERIES EDITOR'S PREFACE

'Et moi, ..., si j'avait su comment en revenir,
je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be
able to do something with it.

O. Heavyside

One service mathematics has rendered the
human race. It has put common sense back
where it belongs, on the topmost shelf next
to the dusty canister labelled 'discarded non-
sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the *raison d'être* of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

"Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the

extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p -adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five sub-series: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

The algebras from the title of the book are the algebras of analysis; that is (partly) those algebraic objects that naturally come up in various analytic investigations. Even so, the homology and cohomology of these objects might seem pretty far removed from the more daily concerns of analysis. Let me try to indicate why that is not really the case.

To a very large extent, I submit - the author is more cautious in his own introduction - that homology is concerned with (the systematic study of) obstructions to the performing of desired constructions; in particular unavoidable essential obstructions. Homology and cohomology groups are then the natural places for these objects to live in. Thus, any kind of systematic information on these groups (or rings, or sets, or ...), such as they being zero in a certain range, carries a great deal of information on the possibility of performing all kinds of constructions. All of the mathematics is full of constructions - with analysis by no means an exception - and, thus, homology and cohomology have their place in just about all parts of mathematics, though certainly the use of these techniques is more advanced in some.

As the author remarks, the subject of the (co)homology of the algebras of analysis has reached a certain plateau of maturity, making this a natural time for a comprehensive text on the topic. The original Russian version received the prize for the best book published by a university press in 1988, and that is another reason why I am happy to submit this translation to the scrutiny of the English reading part of the mathematical world.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Светлой памяти моих дорогих учителей

Марка Ароновича Наймарка

и

Георгия Евгеньевича Шилова

In fond memory of my dear, late teachers

Mark Aronovich Naimark

and

Georgii Evgenyevich Shilov.

TRANSLATOR'S NOTE

It would have been impossible to make this translation without the help of my wife who typed all the manuscript using the ChiWriter word-processor and helped me with the Russian grammar.

I am also grateful to my colleagues, Professor H.G. Dales and Dr. J.C. McConnell, for their collaboration. Their expert advice was invaluable in clarifying the text and pointing out errors.

University of Leeds
January, 1989.

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PREFACE TO THE ENGLISH EDITION

I am very pleased that this book, which had been initially published in Russian, will now be available to the English reader. The initiative of its translation by Kluwer Academic Publishers belongs to Professor Hazewinkel, to whom I wish to express my most cordial gratitude.

I also wish to express my sincere gratitude to the translator, Dr. A. West, for his efforts and high professional skill.

After the work of the translator had been completed, some of my friends and colleagues offered me invaluable and absolutely unselfish aid in editing the English text. Most of all I am grateful to Professor H.G. Dales (Leeds), as an accomplished mathematician and as a man of irreproachable feeling for the "correct English phrase". His typically English sense of humour, which helped us out in moments of fatigue, also played an important role.

Work on the text took place when both of us were guests of the Centre of Mathematical Analysis at the Australian National University. After leaving, Professor Dales put this work into the reliable hands of Professors G.R. Allan (Cambridge), R.J. Loy (Canberra) and N. Groenbaek (Copenhagen). With the aid of this most effective team, work on editing the book was completed.

Thus if the book will indeed interest English speaking specialists, this will be due to a great extent to the merits of all the mathematicians mentioned above, but the responsibility for any shortcomings must, of course, be borne by the author.

Successful editing was due to the stimulating atmosphere at the CMA and at the Department of Mathematics of the ANU. I express my thanks to their staff, especially to Professor Loy, whom I have already mentioned, Professor I. Kluvanek, Senior Research Fellow of CMA, and Mrs. M. Gray, Secretary of the CMA, for their cordial hospitality. The persons mentioned, and also Professor G. Brown, my generous host at the University of New South Wales, were responsible for the fact that this page of my life, which has already been turned over, was so successful both in the professional and human sense.

... In recent years the homological theory of Banach and topological algebras is more often referred to as topological homology - following Professor M. Putinar (Bucharest). During the three years following the publishing of the Russian edition of this book, the field has continued to develop. The number of followers both in the East and the West has grown, enlarged by a group of young mathematicians, while we, "the old fellows", still continue, grumbling, to scratch our pens. There, some of my colleagues have expressed the

opinion that this edition should reflect, if only in the most concise and conspactive form, some of the latest results closely related to the material of the book. The short "postscriptum" accompanying the English translation of the book serves this purpose. It is, in a way, a guide to new works. I am grateful to the publishers and personally to Professor Hazewinkel, who have cordially agreed to include this additional text in the edition now being prepared.

ALEXANDER HELEMSKII

Moscow
March 1989