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Interpolation Theory and Its Applications

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Introduction

1. Interpolation problems play an important role both in theoretical and applied investigations. This explains the great number of works dedicated to classical and new interpolation problems ([1]–[5], [8], [13]–[16], [26]–[30], [57]).

In this book we use a method of operator identities for investigating interpolation problems. Following the method of operator identities we formulate a general interpolation problem containing the classical interpolation problems (Nevanlinna–Pick, Caratheodory, Schur, Hamburger, Krein) as particular cases. We write down the abstract form of the Potapov inequality. By solving this inequality we give the description of the set of solutions of the general interpolation problem in the terms of the linear–fractional transformation. Then we apply the obtained general results to a number of classical and new interpolation problems.

Some chapters of the book are dedicated to the application of the interpolation theory results to several other problems (the extension problem, generalized stationary processes, spectral theory, nonlinear integrable equations, functions with operator arguments).

2. Now we shall proceed to a more detailed description of the book contents.

Let G and H be Hilbert spaces. The symbol $\{H_1, H_2\}$ denotes the set of all the bounded linear operators acting from H_1 to H_2 . Let us consider the set of the operators $A, S, B \in \{H, H\}$ and $\Pi, Q \in \{G, H\}$ satisfying the relation

$$AS - SB = \Pi \cdot Q^* \quad (0.1)$$

Identity (0.1) is a generalization of the commutation relation. Let us single out the case in which the space G is even–dimensional, i.e.

$$G = G_1 \oplus G_1, \quad \dim G_1 = n < \infty \quad (0.2)$$

According to decomposition (0.2) the operators Π and Q have the following block representation

$$\Pi = [\Phi_1, \Phi_2], \quad Q = [\Psi_1, \Psi_2]; \quad \Phi_k, \Psi_k \in \{G_1, H\}$$

Identity (0.1) now takes the form

$$AS - SB = \Phi_1 \Psi_1^* + \Phi_2 \Psi_2^* \quad (0.3)$$

If

$$\Psi_1 = -i \Phi_2, \quad \Psi_2 = -i \Phi_1, \quad B = A^*, \quad (0.4)$$

then (0.3) implies the identity

$$AS - SA^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*) \quad (0.5)$$

The identities of form (0.1), (0.3), (0.5) play an important role in a number of analysis problems [53], [55], [56]. In this book the method of operator identities is used when investigating interpolation problems.

3. Let us present the main examples of the operator identities used in the book.

Example 1 In the space of vector-valued functions

$$f(t) = \text{col} [f_1(t), f_2(t), \dots, f_n(t)],$$

let us consider operators of the form

$$Sf = \frac{d}{dx} \int_0^\omega s(x, t) f(t) dt, \quad (0.6)$$

where the matrix kernel $s(x, t)$ has the following structure

$$s(x, t) = \left\{ s_{l,m}(\omega_l x - \omega_m t) \right\}_{l,m=1}^n; \quad \omega_k = \overline{\omega}_k, \quad \omega_k \neq 0. \quad (0.7)$$

We assume that the operator S is bounded on the space of vector-valued functions $L_n^2(0, \omega)$ and that $s_{l,m}(v) \in L^2(-\omega_m \omega, \omega_l \omega)$. Let us set further

$$\begin{cases} M(x) = Ws(x, 0), \\ N(x) = -Ws(0, x), \end{cases} \quad 0 \leq x \leq \omega \quad (0.8)$$

where

$$W = \text{diag} \{ \omega_1, \omega_2, \dots, \omega_n \}.$$

By the direct calculation we obtain the relation

$$(AS - SA^*)f = i \int_0^\omega [M(x) + N(t)] f(t) dt. \quad (0.9)$$

Here A has the form

$$Af = iW \int_0^x f(t) dt \quad (0.10)$$

Let G_1 be the Euclidean space of vector-columns g of dimension n . Let us introduce the following operators acting from G_1 to $L_n^2(0, \omega)$:

$$\begin{aligned} \Phi_1 g &= M(x)g, & \Phi_2 g &= g, \\ \Psi_1 g &= -ig, & \Psi_2 &= -iN^*(x)g \end{aligned} \quad (0.11)$$

Using the notation (0.9) and the fact that $B = A^*$, we can rewrite (0.9) in the form of the operator identity (0.3). If $M(x) = N^*(x)$, then we get identity (0.5). Let us note that for $\omega_1 = \omega_2 = \dots = \omega_n$ an operator S of form (0.6), (0.7) is an operator with a difference kernel $s(x, t) = s(x - t)$. If $\omega_1 = \omega_2 = \dots = \omega_p = a$,

$\omega_{p+1} = \omega_{p+2} = \dots = \omega_n = -a$ ($1 \leq p \leq n$), then the kernel $s(x, t)$ has the following structure

$$s(x, t) = \begin{bmatrix} s_{11}(x-t) & s_{12}(x+t) \\ s_{21}(x+t) & s_{22}(x-t) \end{bmatrix} \} p \text{ rows} \quad (0.12)$$

Example 2 Let matrices $v_1, v_2, \dots, v_n$ of the order $m \times m$ and points $z_1, z_2, \dots, z_n$ ($\text{Im } z_k > 0$) be given. Let us consider the matrices

$$A = \text{diag} \{z_1 E_m, z_2 E_m, \dots, z_n E_m\}, \quad (0.13)$$

$$\Phi_1 = -i \text{col} [v_1, v_2, \dots, v_n], \quad (0.14)$$

$$\Phi_2 = \text{col} [E_m, E_m, \dots, E_m], \quad (0.15)$$

Putting

$$S = \left[\frac{v_j - v_k^*}{z_j - \bar{z}_k} \right]_{j,k=1}^n \quad (0.16)$$

we deduce operator identity (0.5).

Example 3 Let $S_0, S_1, \dots, S_{2n-2}$ be matrices of order $m \times m$. Let us introduce the following matrices

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ E_m & 0 & \dots & 0 & 0 \\ 0 & E_m & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E_m & 0 \end{bmatrix}; \quad \Phi_1 = -i \begin{bmatrix} 0 \\ S_0 \\ S_1 \\ \dots \\ S_{n-2} \end{bmatrix}; \quad \Phi_2 = \begin{bmatrix} E_m \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \quad (0.17)$$

$$S = \begin{bmatrix} S_0 & S_1 & \dots & S_1 \\ S_1 & S_2 & \dots & S_n \\ \dots & \dots & \dots & \dots \\ S_{n-1} & S_n & \dots & S_{2n-2} \end{bmatrix} \quad (0.18)$$

Relation (0.5) holds again.

4. The first chapter of the book is devoted to the interpolation problem generated by operator identities of the form (0.5):

It is required to describe the set of all nondecreasing operator-valued functions $\tau(u)$ with values in the class $\{G_1, G_1\}$ such that the following relations hold

$$S = \int_{-\infty}^{+\infty} (E - Au)^{-1} \Phi_2 [d\tau(u)] \Phi_2^* (E - A^*u)^{-1} + FF^*, \quad (0.19)$$

$$\Phi_1 = -i \int_{-\infty}^{+\infty} \left[A(E - Au)^{-1} + \frac{u}{1+u^2} E \right] \Phi_2 d\tau(u) + i(\Phi_2 \alpha + F\beta^{1/2}), \quad (0.20)$$

where $\alpha, \beta \in \{G_1, G_1\}$, $\alpha = \alpha^*$, $\beta \geq 0$, and the operator F from $\{G_1, H\}$ is defined by

$$AF = \Phi_2 \beta^{1/2}. \quad (0.21)$$

(The integrals on the right-hand sides of (0.19), (0.20) converge in the weak sense).

In many important problems we have $F = 0$, $\beta = 0$. In this case the formulas (0.19), (0.20) take the form

$$S = \int_{-\infty}^{+\infty} (E - Au)^{-1} \Phi_2 [d\tau(u)] \Phi_2^* (E - A^*u)^{-1}, \quad (0.22)$$

$$\Phi_1 = -i \int_{-\infty}^{+\infty} \left[A(E - Au)^{-1} + \frac{u}{1+u^2} E \right] \Phi_2 d\tau(u) + i \Phi_2 \alpha, \quad (0.23)$$

Formula (0.20) directly implies that the inequality

$$S \geq 0$$

is a necessary condition for the interpolation problem stated above to be solvable. The problem is called *non-degenerate* if the following stronger inequality holds

$$S \geq \delta E, \quad \delta > 0.$$

For $S = E$ identity (0.5) takes the form

$$A - A^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*),$$

and formula (0.22) can be written as

$$E = \int_{-\infty}^{+\infty} (E - Au)^{-1} \Phi_2 [d\tau(u)] \Phi_2^* (E - A^*u)^{-1}.$$

Let us note that the interpolation problem considered by Louis de Branges [8] is related to the case $S = E$.

Let $\tau(u)$, α, β be a solution of the interpolation problem stated above. Let us associate with this solution the Nevanlinna function

$$v(z) = \beta z + \alpha + \int_{-\infty}^{+\infty} \left(\frac{1}{u-z} + \frac{u}{1-u^2} \right) d\tau(u) \quad (0.24)$$

Operator identity (0.5) and problem (0.19), (0.20) will be associated with the inequality

$$L(z) = \begin{bmatrix} S & B(z) \\ B^*(z) & C(z) \end{bmatrix} \geq 0, \quad (0.25)$$

where

$$B(z) = (E - Az)^{-1} [\Phi_1 - i \Phi_2 v(z)], \quad (0.26)$$

$$C(z) = [v(z) - v^*(z)]/(z - \bar{z}) \quad (0.27)$$

The function $v(z)$ in (0.24) is connected with the solution of interpolation problem (0.19), (0.20) by relation (0.25). Inequality (0.25) is an abstract analog of the V. P. Potapov inequalities [11]. In the non-degenerate case inequality (0.25) can be solved explicitly, i.e. a description of the functions $v(z)$ of the form (0.24) satisfying inequality (0.25) can be given. Under some additional assumptions the following assertions are proved:

I. *An operator-valued function $v(z)$ is a solution of (0.25) if and only if it admits the following representation*

$$v(z) = i[a(z)\mathcal{P}(z) + b(z)\mathcal{Q}(z)][c(z)\mathcal{P}(z) + d(z)\mathcal{Q}(z)]^{-1}, \quad (0.28)$$

where the matrix of coefficients is defined as follows

$$\mathfrak{A}(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = E - iz\Pi^*(E - A^*z)^{-1}S^{-1}\Pi J, \quad (0.29)$$

$$J = \begin{bmatrix} 0 & E_1 \\ E_1 & 0 \end{bmatrix}.$$

E_1 is the identity operator on G_1 , and the meromorphic operator-valued functions, $\mathcal{P}(z)$, $\mathcal{Q}(z)$ with values in $\{G_1, G_1\}$ satisfy

$$\det[\mathcal{P}^*(z)\mathcal{P}(z) + \mathcal{Q}^*(z)\mathcal{Q}(z)] \neq 0, \quad \operatorname{Im} z > 0; \quad (0.30)$$

$$\mathcal{P}^*(z)\mathcal{Q}(z) + \mathcal{Q}^*(z)\mathcal{P}(z) \geq 0, \quad \operatorname{Im} z > 0. \quad (0.31)$$

II. *An increasing operator-valued function $\tau(u)$ and $\alpha = \alpha^*$, $\beta \geq 0$ give a solution of interpolation problem (0.19), (0.20) if and only if $v(z)$ of the form (0.24) is a solution of (0.25).*

Chapter I contains solvability conditions for the interpolation problem in the degenerate case as well.

Assertions I, II are used for solving a number of well-known as well as new interpolation problems.

5. In the case of Example I when the operator S has form (0.6), (0.7), interpolation problem (0.19), (0.20) is equivalent to the representation

$$s(x, t) = s(0, t) + \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} (Wu)^{-1} (e^{ixWu} - E) [d\tau(u)] (e^{-iWtu} - E) (Wu)^{-1}. \quad (0.32)$$

If $n = 1$, then $s(x, t)$ is of the form $s(x, t) = s(x - t)$. Let $s(x)$ have a continuous derivative. Then (0.32) implies the well-known M.G.Krein representation [31]

$$k(x) = \int_{-\infty}^{+\infty} e^{ixu} d\tau(u), \quad k(x) = s'(x). \quad (0.33)$$

In the case of Example 2, when relations (0.13)–(0.16) hold, interpolation problem (0.19), (0.20) can be reduced to the truncated Nevanlinna–Pick problem: to describe the set of matrix-valued functions of form (0.24) satisfying the condition

$$v\left(\frac{1}{z_k}\right) = -v_k.$$

In the case of Example 3, when relations (0.17), (0.18) hold, interpolation problem (0.19), (0.20) can be reduced to the Hamburger moment problem: to describe the set of monotone increasing matrices $\tau(u)$ and matrices α such that

$$\begin{aligned} S_k &= \int_{-\infty}^{+\infty} u^k d\tau(u), & 0 \leq k \leq 2n-2, \\ \alpha &= \int_{-\infty}^{+\infty} \frac{u}{1+u^2} d\tau(u). \end{aligned} \tag{0.34}$$

Formulas (0.32), (0.33) giving the solution of the interpolation problem at the same time allow to solve the extension problem, i.e. they permit to extend $s(x, t)$ onto the domain $0 \leq x, t \leq l$ ($l > \omega$) preserving the kernel structure and its Hermitian-positivity. In a similar way formula (0.34) allows to define S_k when $k > 2n-2$.

In Chapter 2 we construct a general scheme for investigating interpolation problems in unit circle. The results of this chapter can be deduced from the results of Chapter 1 with help of the linear fractional transformation. Independent consideration of the interpolation problems in the unit circle makes it possible to do without some restrictions of Chapter 1. These restrictions are connected with the possibility of a jump of the distribution function $\tau(u)$ in the singular point $u = \infty$.

Chapter 3 is dedicated to the Hermitian-positive functions of several variables.

The Hermitian-positive functions play an essential role in a number of analysis problems (probability theory [47], inverse spectral problems [53]). In many of these problems it becomes necessary to extend the Hermitian-positive functions onto a greater domain with the preservation of Hermitian positiveness. For the Hermitian-positive functions of one variable the extension problem was formulated and solved in the classical works by M. Krein [32].

The extension problem of Hermitian-positive functions of two and more variables has been investigated much less (see [11], [48], [52]).

The existence of the non-extendible Hermitian-positive functions was proved in the famous works by Calderon, Pepinsky [11] and Rudin [48]. Concrete examples of non-extendible functions have been built in the author's article [52]. The results of article [52] are described and developed in Chapter 3. The obtained results allow us to replace existence theorems by concrete examples in a number of other problems as well.

Our approach is based on the consideration of the block Toeplitz matrices

$$A = \begin{bmatrix} C_0 & C_1 & \cdots & C_N \\ C_1^* & C_0 & \cdots & C_{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ C_N^* & C_{N-1}^* & \cdots & C_0 \end{bmatrix} \quad (0.35)$$

the blocks C_k ($0 \leq k \leq N$) of which are also Toeplitz matrices. The problem of the extension of the block Toeplitz matrix can be easily solved by the methods of the interpolation theory. However the requirement for the blocks C_k to be Toeplitz ones created a new situation. The analysis of this situation is contained in Chapter 3.

In Chapter 4 we investigate the spaces of the entire functions $H(E)$ (de Branges spaces). De Branges stated and solved the problem of describing measures in the Parseval equality for the spaces $H(E)$ [8]. We put the space $H(E)$ in correspondence with a certain operator identity. It turns out that the corresponding interpolation problem coincides with the Parseval equality problem in the space $H(E)$. In our exposition we essentially follow the preprint by L.Golisky and I.Mikhailova (edited by V.Potapov) [19]. The operator A in the constructed operator identity is a functional model of the operator defined by the de Branges characteristic matrix function. This fact allows us to deduce new theorems of operators linear similarity.

Chapter 5 is dedicated to the case when the matrix S satisfies identity (0.5) and is degenerate, i.e.

$$\det S = 0 \quad (0.36)$$

Canonical representations of the matrix S are deduced for case (0.36). The obtained results extend and compliment the well-known results by A.Markov and M.Krein–A.Nudelman (see [34]) connected with the Hankel matrices. In this chapter we also give a method of solving the V.Potapov inequality for case (0.36). Under additional conditions it is proved that the constructed solution is rational and unique. This fact is essential from the view point of applied problems (extremal problems).

In Chapter 6 we continue illustrating the possibilities of the operator identities method. In section 6.1 this method is used to investigate the Nevanlinna–Pick tangential problem (see [43]). In section 6.2 we consider the operator identity of the form

$$S - ASA^* = \Phi_2 \Phi_2^* - \Phi_1 \Phi_1^* \quad (0.37)$$

We use the results of section 6.2 for analysing the Schur problem (section 6.3) and tangential Nevanlinna–Pick problem in the circle (section 6.4). There are very many works dedicated to the above-mentioned problems [2], [29], [34]. In the last section of Chapter 6 a new class of generalized stationary processes is introduced. For this class we have formulated and solved the generalized M.Krein extension problem.

In Chapter 7 extremal problems of two types are investigated. First we consider the interpolation problems the solution of which $w(\zeta)$ satisfies in addition the

extremal condition

$$w^*(\zeta) w(\zeta) \leq \rho_{\min}^2, \quad |\zeta| < 1 \quad (0.38)$$

We reduce the corresponding interpolation problem to a degenerate one. This degenerate problem has a unique solution which can be found with the help of the results of Chapter 5. It is essential both from the applied and theoretical view-points that the solution of the extremal problem turns out to be a rational matrix function. The case when $\rho_{\min}$ is a scalar matrix was investigated in the works [1], [2] and found its application in the control theory [28]. The transition to the arbitrary non-negative matrix $\rho_{\min}$ allows to increase considerably the class of the extremal problems which have effective solutions.

I am very grateful to Professor P. Fuhrmann for his having called my attention to the importance of the extremal problems and for having advised me to use the operator identity method when investigating these problems.

The second type of the extremal problems considered in Chapter 7 is connected with the maximum jump theorem (A. Sakhnovich [50]). This case includes some problems of the canonical differential systems theory, several problems of radio technique and the problem connected with the Gauss model (Vladimirov-Volovich problem [63]).

In Chapter 8 the following system of difference equations is considered

$$W(k, z) - W(k-1, z) = i z J q(k) W(k-1, z), \quad k > 0 \quad (0.39)$$

where $W(k, z)$, $q(k)$ are matrices of the $(2m) \times (2m)$ order and

$$q(k) \geq 0, \quad q(k) J q(k) = 0, \quad W(0, z) = E_{2m} \quad (0.40)$$

Problems of interpolation theory, theory of integrating Toda chain equation [54], theory of non-selfadjoint operators [36] lead to the systems of form (0.39), (0.40). Systems of difference equations generated by the Jacobi matrices [6] can be reduced to system (0.39), (0.40).

In Chapter 8 the direct and inverse spectral problems for systems (0.39), (0.40) are studied. System (0.39), (0.40) is a discrete analogue of the system

$$\frac{dW(x, z)}{dx} = i z J q(x) W(x, z), \quad x \geq 0 \quad (0.41)$$

the spectral theory of which was investigated in the papers [53], [55]. With the help of the results obtained for system (0.39) on the half-axis ($k > 0$) the spectral problem on the axis ($-\infty < k < \infty$) is analysing too. As in the classical case (Sturm-Liouville equation) when passing from the problem on the axis to the corresponding problem on the half-axis the system dimension is doubled.

In Chapter 9 the method of inverse spectral problems is applied to the analysis of the nonlinear integrable equations (discrete case). The law of the spectral data

evolution is deduced and the hierarchy of the equations is constructed in this chapter. When $N = 1$ the member of the hierarchy coincides with the Toda equation, when $N = 2$ the member of the hierarchy coincides with the Volterra equation.

We should like to note that Yu. Berezanskii started using the method of the inverse spectral problem for the investigation of the nonlinear discrete systems [7].

In Chapter 10 the evolution law of spectral data is deduced for one nonlinear system of differential–difference equations. In particular the well-known Toda equation

$$\frac{d^2 x(k, t)}{dt^2} = \exp[x(k-1, t) - x(k, t)] - \exp[x(k, t) - x(k+1, t)], \quad k \geq 1 \quad (0.42)$$

can be reduced to this system.

Our general theory is applied to the investigation of the important special case when

$$x(0, t) = 0 \quad (0.43)$$

We describe the procedure of constructing the solution in this case. The case of the finite chain with the conditions

$$\begin{cases} x(0, t) = 0, \\ x(N+1, t) = \infty \end{cases} \quad (0.44)$$

is studied separately. It turns out that the solutions in case (0.42), (0.44) can be presented by non-reflecting potentials of the Sturm–Liouville equations.

In Chapter 11 we again consider the operator identities of the type

$$AS - SA^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*) \quad (0.45)$$

$$S - AS A^* = \Phi_1 \Phi_2^* + \Phi_2 \Phi_1^* \quad (0.46)$$

$$S - AS A^* = \Phi_2 \Phi_2^* - \Phi_1 \Phi_1^* \quad (0.47)$$

In Chapter 1 we deduce the formula expressing Φ_1 by Φ_2 (case (0.45)). Similar results were received in Chapter 2 (case (0.46)) and in Chapter 6 (case (0.47)). In Chapter 11 the formulas expressing Φ_1 by Φ_2 are interpreted in a new way. It is proved that these formulas can be written in the form

$$\Phi_1 = \Psi(B) \Phi_2 \quad (0.48)$$

where B is an operator argument of the matrix function $\Psi(z)$ of the $m \times m$ order. With the help of representation (0.48) we have managed to transfer partially the famous results by Sarason [57] concerning commuting operators from the scalar case

($m = 1$) on the matrix case ($m > 1$). We also managed to deduce the formulas permitting to factorize effectively the operators of a number of classes. By factorization we understand the representation of the operator S in the form

$$S = S_- S_-^*, \quad S_- = \Psi(B) \quad (0.49)$$

The representation of the operator S in form (0.49) is called by us B -factorization of this operator.

Let us note in concrete examples B -factorization usually implies the classical triangular factorisation.