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Fractional Calculus for Scientists and Engineers

 Springer

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To the memory of my parents

*José Ortigueira and Rita Rosa da Ascensão
Duarte*

that did their best for me.

To my children

*Eduardo and Joana, and my wife, Laurinda,
for their love and patience.*

Preface

Fractional Calculus has been attracting the attention of scientists and engineers from long time ago, resulting in the development of many applications. Since the nineties of last century fractional calculus is being rediscovered and applied in an increasing number of fields, namely in several areas of Physics, Control Engineering, and Signal Processing. However, most of the theoretical setup of Fractional Calculus was done by mathematicians that directed their attention preferably to the so-called Riemann-Liouville and/or Caputo derivatives. We must remark that most of the articles that appear in the scientific literature, in the framework of the fractional calculus and their applications, the authors use those derivatives but at the end they contrast their model using a numerical approach based in a finite number of terms from the series that define the Grünwald-Letnikov derivative. This may be confirmed in several books that appeared recently and is one justification for the present one. It intends to present a Fractional Calculus foundation based of the Grünwald-Letnikov derivative, because it exhibits great coherence allowing us to deduce from it the other derivatives, which appear as a consequence of the Grünwald-Letnikov derivative properties and not as a prescription. The Grünwald-Letnikov derivative is a straight generalisation of the classic derivative and leads to formulae and equations that recover the classic ones when the order becomes integer.

On the other hand, the available literature deals mainly with the causal (anti-causal) derivatives. In situations where no preferred direction exists it is common to use the Riesz potentials. Alternatively we will present the two-sided fractional derivatives that are more general than the Riesz potentials and are generalizations of the classic symmetric derivatives. These allow us to deal comfortably with fractional partial differential equations. Similarly, the Quantum Derivative is presented as a useful tool for dealing with the fractional Euler-Cauchy equations that are suitable for dealing with scale invariant systems. These derivatives are not described in published books.

This book is directed towards Scientists and Engineers mainly interested in applications who do not want to spend too much time and effort to access to the main Fractional Calculus features and tools. For this reason readers can “jump”

the chapter 2 in a first reading. The book is written in a cursive way, like a divulgation text, reducing the formalism to increase the legibility. I hope I have been successful.

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I would like to remember here and thank all the “integer” people I have been meeting at the Workshops and Symposia. Besides those I cite at the introduction I would like to refer, without any special order: Om Agrawal, Raoul Nigmatullin, Yury Luchko, Bohdan Datsko, Mohammad Tavazoei, Vladimir Uchaikin, Yang-Quan Chen, Jocelyn Sabatier, Pierre Melchior, Duarte Valério, Isabel de Jesus, and Francesco Mainardi. Probably I forgot others. I beg their pardon.

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