

# Non-Linear Mechanics of Materials

# SOLID MECHANICS AND ITS APPLICATIONS

Volume 167

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*Series Editor:* G.M.L. GLADWELL

*Department of Civil Engineering  
University of Waterloo  
Waterloo, Ontario, Canada N2L 3G1*

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Jacques Besson • Georges Cailletaud •  
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# Non-Linear Mechanics of Materials

In cooperation with Marc Blétry

 Springer

Jacques Besson  
Centre des Matériaux  
CNRS UMR 7633  
MINES ParisTech  
BP 87, 91003 Evry Cedex  
France  
[jacques.besson@mines-paristech.fr](mailto:jacques.besson@mines-paristech.fr)

Samuel Forest  
Centre des Matériaux  
CNRS UMR 7633  
MINES ParisTech  
BP 87, 91003 Evry Cedex  
France  
[samuel.forest@mines-paristech.fr](mailto:samuel.forest@mines-paristech.fr)

Georges Cailletaud  
Centre des Matériaux  
CNRS UMR 7633  
MINES ParisTech  
BP 87, 91003 Evry Cedex  
France  
[georges.cailletaud@mines-paristech.fr](mailto:georges.cailletaud@mines-paristech.fr)

Marc Blétry  
Institut de Chimie et des  
Matériaux Paris-Est  
CNRS UMR 7182  
Université Paris XII  
2-8, rue H. Dunant  
94320 Thiais  
France  
[bletry@icmpe.cnrs.fr](mailto:bletry@icmpe.cnrs.fr)

Jean-Louis Chaboche  
Dépt. Matériaux et Structures  
Métallique  
Office National d'Études et de  
Recherches Aérospatiales  
(ONERA)  
29 avenue de la Division  
Leclerc  
92322 Chatillon CX  
France  
[jean-louis.chaboche@onera.fr](mailto:jean-louis.chaboche@onera.fr)

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# Foreword

Invited by Jacques Besson, Georges Cailletaud, Jean-Louis Chaboche and Samuel Forest to introduce their book “Nonlinear mechanics of materials”, I am pleased to deliver an affirmative response to the honor, the confidence and the friendship that they thus show me.

- Pleased at an emotional level first, because the authors belong to a scientific community that has blossomed in France for the last twenty years and that has brought much to all of us. Moreover, two of them have been my students and are now friends and colleagues. This community of researchers in solid mechanics, first built on young people, is now mature and reaches out across France and overseas. The strong French participation in mechanical conferences and publications in great scientific journals testify to their contributions. So does this book.
- Pleased at a scientific level, as it is a modern presentation of materials mechanics, from microscopic to mesoscopic scale, from physical to numerical aspects, from phenomenology to homogenization, from field calculations to localization. Everything is in it, or almost! “Nonlinear” is the central point and everybody knows that there are many pitfalls. The authors managed to avoid them thanks to an exhaustive but well delimited approach.
- Pleased, lastly, on a technical level because through the very design of the book, the writers have chosen to address an informed but not expert audience. The elementary notions have indeed been left aside, but to the benefit of a clear synthesis of the developments made in the last twenty years. However, the basic tools called “General concepts” are rigorously described in a concise manner, in spite of the scope of the subjects: virtual power, thermodynamics, solving of nonlinear systems, integration of differential equations, finite element method. The authors, through their education and their research at the Centre des Matériaux of Mines ParisTech and at ONERA use, with the same virtuosity, constitutive equations and numerical calculations; that is the originality and the appeal of this book. Plasticity and viscoplasticity are explained up to multimechanism models; damage up to the problem posed by its deactivation in compression,

so hard to define in 3D; the behavior of heterogeneous materials up to non-linear self-consistent schemes; finite strain (in an accessible manner!) up to Cosserat continuum; structural analysis written for constitutive equations “programmers”, who will find here good “seedlings”; and, at last, the extreme limit of the behavior represented by localization phenomena.

More physics, more numerics reflect well the current trends in Mechanics. No doubt that PhD students, researchers in solid mechanics, engineers close to calculation codes related to optimal use of materials will find in this book matter to deepen their knowledge, many “ready to use” tools and almost 300 references. Jacques, Georges, Jean-Louis, Samuel, thank you for sharing with us your knowledge.

Cachan, 7th May 01

Jean Lemaitre  
Emeritus Professor at Paris-6 University  
LMT-Cachan

# Preface

“Everything should be made as simple as possible, but no simpler”, said a famous swiss-german author. Several tens years after this quote, a lot remains to be done in the realm of materials. In their daily practice, engineers use too often rather naive models. However, significant progress have been made in a recent period. The 80’s have seen the boom of many models, the 90’s the appearance of robust algorithms allowing their use in structural calculations. At the dawn of the new millenary, we are then prepared for a significant overhaul of the design methods in materials mechanics. The remaining is a problem of data, and this is why characterizing is more than ever important. It is mandatory to complete the handbooks, by going way further than the usual information such as yield stress, creep or crack behaviour, by including data on the actual cyclic behaviour, the evolution of the behaviour with microstructure in use, the mechanical response at the scale of phases, etc. One should not neglect education, as the domain of nonlinear mechanics holds specific problems. The authors consider that there is a great risk to see insufficiently qualified engineers handling more and more complex numerical tools. It appears to them that, currently, all conditions are met so that this risk takes shape, with the exclusive celebration of a “systemic” approach that leads (there is only 24 hours per day) to reducing the fundamental teaching and, without a doubt, the technical basis of the young graduates.

However, it goes without saying that it is always necessary to revise the way things are set forth, renew and adapt the teaching methods. That is why we have decided to associate with the description of the models, the methods that are necessary to use them. It is then justified to regroup in one book a summary of models that have become classic and an opening towards burning issues. The first version of the present work appeared during a course at IPSI (Institut pour la Promotion des Sciences de l’Ingénieur), in September 1997, with the same title. The conversion from handout to book that followed turned out to be a long and painfull experience, especially near the end, when one must finish the next day and that, clearly, there will remain so many things to write. . .

Of course, one should study the optimizing methods and identification process, experimental data analysis, to bring the model up to an operational state. Of course, it would also be necessary to deal with lifetime prediction methods. Of course, one

should open wider one's arms and treat the multi-physics coupling problems. So, may the readers that might be disappointed by such deficiencies forgive us, better still, may them write to us, so that the most blatant omissions may be corrected in the future if—as we hope—this book is only the first step of a greater project that should lead to a more up-to-date, and more immediately usable presentation of the models and methods in materials mechanics.

The discussion about the existence of God(s) was not dealt with in this book. To be frank, this would have been out of this series topics, and it might have been difficult to homogenize the notations. In a first approximation, the reader should consider the results as independent of the answer to that question. However, when stepping back to contemplate the work, and give to one's keyboard new tasks, beyond the usual thanks to those who live with us every day, the one who brought us to existence, to our previous and coming masters, comes the need to be silent for a few minutes and, to put some humanity into the forthcoming equations, suggest the reader other readings, as new conjugate directions in space.

J. Besson, G. Cailletaud, J.-L. Chaboche, S. Forest—juin 2001

*The first creature resembled a lion, the second was like a calf, the third had a face like that of a human being, and the fourth looked like an eagle in flight.*  
(Revelation 4,7)

*Avec soulagement, avec humiliation, avec terreur, il comprit que lui aussi  
était une apparence, qu'un autre était en train de le rêver.*  
Jorge Luis Borges (Fictions, Les ruines circulaires)

*Die Eswelt hat Zusammenhang im Raum und in der Zeit.  
Die Duwelt hat in Raum und Zeit keinen Zusammenhang.*  
Martin Buber (Ich und Du)

*J'avance en poésie comme un cheval de trait  
tel celui-là de jadis dans les labours de fond  
qui avait l'oreille dressée à se saisir réel  
les frais matins d'été dans les mondes brumeux.*  
Gaston Miron (L'homme rapaillé)



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